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REALIZATION OF COHOMOLOGY CLASSES IN ARBITRARY EXACT CATEGORIES

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Introduction

The language of exact categories allows finite limits and quotients of equivalence relations and has an axiom which states that quotients are preserved by inverse images of maps. This paper shows in detail how such a language is precisely what is needed to formulate the concept of cohomology class in each dimension $n \geq 0$ and to establish the functorial properties of cohomology. Given a particular exact category $\mathscr{C}$, one obtains a family of abelian group valued functors $H^n(\mathscr{C}, -)$ defined on abelian group objects of $\mathscr{C}$. A cohomology class from $H^n(\mathscr{C}, A)$ is realized directly in $\mathscr{C}$ as an algebraic structure called an 'n-torsor', a certain kind of group action of the coefficient group A. There is no intervening construct such as, for example, a resolution.

For an introductory illustration of a '1-torsor', consider the following classical example. Let G be a group and let $\mathscr{S}^G$ denote the (exact) category of G-sets. Given a G-module A, (an abelian group object of $\mathscr{S}^G$), an element of $H^1(G,A)$ is a short exact sequence of G-modules

$$(f,g):0 \to A \to E \to Z \to 0$$

where G acts trivially on Z. In the category of G-modules (which is also exact), the map $f:A \to E$ determines an action of A on E defined by $ya = -fa + y$, $y \in E$, $a \in A$, with two characteristic properties: (1) the map $E \times A \to E \times E$ sending (y,a) to (y, ya) is a monic whose image is an equivalence relation on E and (2) the quotient of this equivalence relation is $g:E \to Z$. These data, summarized in the 'exact diagram' of G-modules

$$E \times A \underset{\text{proj.}}{\overset{\text{action}}{\rightrightarrows}} E \xrightarrow[g]{} Z$$

comprise a '1-torsor under A over Z'.

Since properties (1) and (2) can be interpreted in non-abelian exact categories, one

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may instead consider the same cohomology class of $H^1(G,A)$ as follows. Let $E_0 = \{y \in E \mid gy = 1\}$. The action of A on E given above restricts to an action of A on E_0 in $\mathcal{S}^G$ and has the property that the map sending (y,a) to (y, ya) is an isomorphism of G-sets $E_0 \times A \to E_0 \times E_0$. The corresponding exact diagram $E_0 \times A \rightrightarrows E_0 \to 1$ in $\mathcal{S}^G$ is again a 1-torsor under A over 1. This torsor represents the zero cohomology class iff there exists a G-map $1 \to E_0$ (or, equivalently, G fixes an element of E_0). The group structure of $H^1(G,A)$ correlates the Baer sum of two cohomology classes in the abelian category of G-modules with the Whitney sum of the two corresponding torsors in the exact non-abelian category $\mathcal{S}^G$.

The 1-torsor just considered possesses another important structure – that of a groupoid – where E_0 comprises the vertices, $E_0 \times A$ the edges, and where the groupoid multiplication is defined by $(y,a)(ya,a') = (y, a + a')$. Note the special way in which A is involved, in particular that the projection $E_0 \times A \to A$ is a groupoid homomorphism. In fact, since any groupoid related in this manner to the G-module A determines a 1-torsor whose groupoid it is, one is led to focus, in dimension 1, on groupoids and projection maps ('fibrations') into A.

The concepts of groupoid and fibration can easily be extended so as to yield n-dimensional torsors which represent cohomology classes of H^n. Simplicial algebra is used to accomplish this extension because it yields very concise workable definitions and the easy and natural transition between dimension 1 and dimension $n > 1$ structures. The role played in dimension 1 by groupoids is played in dimension n by an algebraic structure called an 'n-dimensional hypergroupoid' whose structure consists of a kind of generalized composition law satisfying certain equations. Hypergroupoids and hypergroupoid actions (again, fibrations) comprise the technical framework of the entire theory.

In order to realize an n-dimensional cohomology class with coefficients in A, one associates to A a basic kind of n-dimensional hypergroupoid denoted $K(A,n)$. (As the notation suggests, this hypergroupoid is a simplicial Eilenberg–MacLane space.) Cohomology classes are represented by actions of $K(A,n)$ called 'n-torsors'. These actions are characterized by properties analogous to those observed in the example of a 1-torsor. One may then systematically develop the functorial properties of H^n, its group structure, and the long exact sequence.

The axioms for n-dimensional hypergroupoids and n-torsors are really axiom-schema in exact category language with 'n' as the only parameter. In effect, one uniform theory applies in all dimensions. We will prove that every n-torsor is a 1-torsor in the category of $(n-1)$-dimensional hypergroupoids. This reinterpretation again emphasizes the uniformity of the torsor concept by detaching it from dependence on dimension as much as possible: "all torsors are 1-torsors". It also results in simplified proofs of functoriality, etc.

When $\mathcal{C}$ is a category for which cohomology groups can be defined in the traditional manner [4, 14], the question arises of how these groups compare with the ones defined in this paper. The answer, in the case of group cohomology, Ext, sheaf cohomology and many others, is that the groups are isomorphic.

The methods described in this paper serve both to define and concretely realize cohomology classes in categories to which the traditional methods do not apply. Examples and applications in such categories will be discussed in later papers.

I am happy to acknowledge my great debt to J.W. Duskin for his mathematical ideas and advice concerning the results presented here. This paper is based on my thesis [8] which was written under his direction and which took his earlier work on torsors [7] as a starting point. I would also like to thank F.W. Lawvere and S. Schanuel for their help while I was working on my thesis, and J.C. Cole for his helpful remarks leading to a (corrected) proof of Theorem 5.7.5.

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1. Exact categories and simplicial objects

1.1. Definition of exact category

Let $\mathscr{C}$ be a category.

Definitions. The *kernel pair* of a map $p:E\to X$ is a pair of maps $p_0, p_1:R\to E$ such that $pp_0=pp_1$ and such that if $pq_0=pq_1$ then $q_i=p_iu$, $i=0,1$ for a unique map u. A pair of maps $f,g:R\to E$ is an *equivalence pair* if for every T the function $\mathscr{C}(T,R)\to\mathscr{C}(T,E)\times\mathscr{C}(T,E)$ sending h to (fh,gh) is a monomorphism whose image is an equivalence relation on $\mathscr{C}(T,E)$.

Definition. The category $\mathscr{C}$ is *exact* if it has all finite limits, if all its equivalence pairs have coequalizers, and if the pullback of any coequalizer is again a coequalizer.

Note that any kernel pair is an equivalence pair. In any exact category, an equivalence pair is the kernel pair of its coequalizer and a coequalizer is the coequalizer of its kernel pair.

Definition. The diagram $R\rightrightarrows E\to X$ is called *exact* if $E\to X$ is the coequalizer of $R\rightrightarrows E$ and $R\rightrightarrows E$ is the kernel pair of $E\to X$.

In an exact category, any map factors uniquely as a coequalizer followed by a monic: the coequalizer is that of the kernel pair of the map. The composite of coequalizers is a coequalizer and q is a coequalizer if qp is. From here on, 'epimorphism' will be used in place of 'coequalizer'.

Some examples of exact categories are: the category $\mathscr{S}$ of sets, the category $\mathscr{S}^{\mathscr{C}}$ of functors $\mathscr{C}\to\mathscr{S}$, the category Sh($X$) of set-valued sheaves on the topological space X, any topos (in fact), any category monadic over $\mathscr{S}$ (e.g. groups, rings, k-algebras, etc.), any abelian category.

1.2. Yoneda-elements

Let $F:\mathcal{I}\to\mathcal{C}$ be a functor. It follows from Yoneda's lemma that $X\cong\varprojlim_i F(i)$ iff $\mathcal{C}(T,X)\cong\varprojlim\mathcal{C}(T,F(i))$ for all T in $\mathcal{C}$.

Definition. A *Yoneda-element* (or simply *element*) of X is a map $x:T\to X$. Write $x\in X$ (abusing notation).

The following example illustrates how the concept of Yoneda-element will be used. The diagram

$$\begin{array}{ccc} E & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

is a pullback in $\mathcal{C}$ iff for any T,

$$\begin{array}{ccc} \mathcal{C}(T,E) & \longrightarrow & \mathcal{C}(T,A) \\ \downarrow & & \downarrow \\ \mathcal{C}(T,B) & \longrightarrow & \mathcal{C}(T,X) \end{array}$$

is a pullback in $\mathcal{S}$ with the functions induced by composition. If $a:T\to A$ and $b:T\to B$, then $(a,b)\in E$ iff $fa=gb$. Note that $\mathrm{pr}_A(a,b)=a$ and $\mathrm{pr}_B(a,b)=b$. The reference to 'T' in discussing Yoneda-elements may be deleted so long as one understands that 'elements' are morphisms.

If $f:X\to Y$, and $x\in X$, then $fx\in Y$. (Yoneda's lemma implies that every function of Yoneda-elements, actually a natural transformation $\mathcal{C}(-,X)$, $\mathcal{C}(-,Y)$, arises from a morphism $f:X\to Y$). $f:X\to Y$ is a monomorphism iff $fx=fx'$ implies $x=x'$. $gf=h$ iff $gfx=hx$ for all (suitable) x.

1.3. Barr-elements

In [1] and [2] Barr proved an Embedding Theorem one of whose consequences is that for any small exact category $\mathcal{C}$ there is a family $\{F_i\}_I$ of set-valued limit- and epi-preserving functors which are collectively faithful and collectively limit- and epi-reflecting. 'Collectively faithful' means that if $F_i(f)=F_i(g)$ for all i then $f=g$. 'Collectively limit-reflecting' and 'collectively epi-reflecting' have obvious analogous meanings.

Suppose one has a diagram in $\mathcal{C}$ involving finite limits and coequalizers. Applying an arbitrary limit- and epi-preserving $F:\mathcal{C}\to\mathcal{S}$ to the diagram yields a diagram in $\mathcal{S}$ having the same limits and epis (surjections) as the original. As a consequence of Barr's theorem, any conclusion one may come to about this diagram in $\mathcal{S}$ (e.g. that it commutes or that something in it is a limit or a surjection) must hold for the original diagram also since among the arbitrary $F: \mathcal{C} \to \mathcal{S}$ are the F_i of Barr's theorem.

1.4. Simplicial objects

Definition. A *simplicial object* $X_\bullet$ in $\mathcal{C}$ is a collection of objects X_n ($n \geq 0$) together with maps

$$X_n \xleftarrow[d_i]{} X_{n+1} \xrightarrow[s_i]{} X_{n+2}$$

for $i = 0, \ldots, n+1$ which satisfy the following (*simplicial*) identities:

$$\begin{array}{llll} d_i d_j = d_{j-1} d_i & \text{for } i<j, & d_i s_j = s_{j-1} d_i & \text{for } i<j, \\ s_i s_j = s_{j+1} s_i & \text{for } i \leq j, & d_i s_j = s_j d_{i-1} & \text{for } i \geq j+2, \\ & & d_{i+1} s_i = d_i s_i = 1. & \end{array}$$

One may visualize $x \in X_n$ (an *n-simplex*) as an n-dimensional polyhedron with vertices $v_0, \ldots, v_n$. In that case, $d_i x \in X_{n-1}$ is the polyhedron spanned by $v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ (the 'face opposite v_i'). The simplicial identities '$d_i d_j$' (*face maps*) are faces-of-faces incidence relations. The equations involving 's_i' (*degeneracies*) do the same for 'degenerate' polyhedra.

Definition. A *simplicial map* $f_\bullet : X_\bullet \to Y_\bullet$ is a family $f_n : X_n \to Y_n$ ($n \geq 0$) which commutes with all the d_i's and s_j's.

The category of simplicial objects in $\mathcal{C}$ is denoted Simpl($\mathcal{C}$).

Definition. An *augmented simplicial object,* denoted $X_\bullet \to X$, is a simplicial object $X_\bullet$ together with a map $p: X_0 \to X$ such that $pd_0 = pd_1$. A *simplicial map* between $X_\bullet \to X$ and $X'_\bullet \to X$ is a simplicial map $f_\bullet$ such that $p' f_0 = p$.

1.5. *n-truncation, n-th simplicial kernel and* COSK^n

Definition. An *n-truncated* simplicial object, denoted $X_{\bullet,\mathrm{tr}}$, consists (only) of $X_0, \ldots, X_n$ and the usual face and degeneracy maps between these.

The process of n-truncating is a functor. If $\mathcal{C}$ has finite limits, then a right adjoint denoted cosk^n exists and can be described using the concept of 'simplicial kernel'.

Definition. Let $n > 1$. The *n-th simplicial kernel* of $X_\bullet$ is an object denoted $\Delta^\bullet(n)(X_\bullet)$ together with maps $p_i : \Delta^\bullet(n)(X_\bullet) \to X_{n-1}$, $i = 0, \ldots, n$, universal with respect to satisfying $d_i p_j = d_{j-1} p_i$ for all $i < j$.

An element of $\Delta^\bullet(n)(X_\bullet)$ is $(x_0, \ldots, x_n)$ where $x_i \in X_{n-1}$, $d_i x_j = d_{j-1} x_i$ for all $i<j$ and $p_i(x_0, \ldots, x_n) = x_i$. It may be visualized as a collection of $(n-1)$-simplices whose faces match so as to form a 'hollow' n-simplex.

The projections p_i play the role of face maps. Using the simplicial identities one may define $q_j : X_{n-1} \to \Delta^\bullet(n)(X_\bullet)$, $0 \le j \le n-1$, which play the role of degeneracies, e.g. $q_0 x = (x, x, s_0 d_1 x, \ldots, s_0 d_{n-1} x)$. Thus, if one begins with an n-truncated simplicial object $X_{\bullet,\mathrm{tr}}$ one may build up a new simplicial object by iterating the simplicial kernel construction (starting at dimension $n+1$). The result is denoted $\operatorname{cosk}^n(X_{\bullet,\mathrm{tr}})$.

The functor $\mathrm{Simpl}(\mathscr{C}) \to \mathrm{Simpl}(\mathscr{C})$ obtained by truncating to dimension n and then applying cosk^n is denoted COSK^n. The assertion $X_\bullet \cong \mathrm{COSK}^n(X_\bullet)$ is a brief way of saying that X_m is a simplicial kernel for all $m > n$.

The canonical projection $X_n \to \Delta^\bullet(n)(X_\bullet)$ sending x to $(d_0 x, \ldots, d_n x)$ need not be epic. If it is epic, $X_\bullet$ is said to be *aspherical at dimension n.* Complexes which are aspherical at all dimensions are called *aspherical.*

1.6. *Vector and matrix notation*

Suppose $X_\bullet \cong \mathrm{COSK}^n(X_\bullet)$. Denote the $(n+1)$-simplex $(x_0, \ldots, x_{n+1})$ by $\boldsymbol{x}$. An $(n+2)$-simplex consists of a sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+2})$ which may be organized as a matrix $[x_{ij}]$, $0 \le i \le n+2$ and $0 \le j \le n+1$, whose i-th row is $\boldsymbol{x}_i$. The simplicial identities $d_i d_j = d_{j-1} d_i$ for $i<j$ determine a pattern in the entries of $[x_{ij}]$ namely: $x_{ji} = x_{i,j-1}$. Here, for example, is a 3-simplex in $\mathrm{COSK}^1(X_\bullet)$:

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0 & x_1' & x_2' & x_3' \\ x_1 & x_1' & x_2'' & x_3'' \\ x_2 & x_2' & x_2'' & x_3''' \\ x_3 & x_3' & x_3'' & x_3''' \end{bmatrix}$$

Note that any row is completely determined by the other rows.

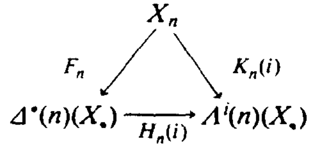
1.7. *Open i-horns and Kan complexes*

Definition. Given $X_\bullet, n>1$ and $0 \le i \le n$, denote by $\Lambda^i(n)(X_\bullet)$ the object universal with respect to having projections $p_j : \Lambda^i(n)(X_\bullet) \to X_{n-1}$ for $0 \le j \le n$ and $j \ne i$ satisfying $d_j p_k = d_{k-1} p_j$ for $j<k$, $j,k \ne i$.

An element of $\Lambda^i(n)(X_\bullet)$ is, in effect, a 'hollow' n-simplex whose face opposite v_i is 'missing': hence the term 'open i-horn' for an element of $\Lambda^i(n)(X_\bullet)$.

If the map $X_n \to \Lambda^i(n)(X_\bullet)$sending x to $(d_0 x, \ldots, d_{i-1} x, -, d_{i+1} x, \ldots, d_n x)$ is epic for each $i = 0, \ldots, n$, then $X_\bullet$ satisfies the *Kan extension condition* at dimension n. If this map is epic for all n, $X_\bullet$ is called a *Kan complex.*

Given $X_\bullet$ consider the diagram of canonical maps:



Lemma 1.7.1. *F_n epic implies $H_{n+1}(i)$ epic.*

Proof. We will apply Barr's Embedding Theorem and prove the lemma in $\mathscr{S}$. Given $(x_0, \dots, -, \dots, x_{n+1}) \in \Lambda^i(n)(X_{\bullet})$ we must find $x \in X_n$ such that $(x_0, \dots, x, \dots, x_n) \in \Delta^{\bullet}(n+1)(X_{\bullet})$. Since the faces of such an x are determined by the x_j's (i.e. $d_k x = d_{i-1} x_k$ for $k<i$ and $d_k x = d_i x_{k+1}$ for $k \geq i$) we have $(d_0 x, \dots, d_n x) \in \Delta^{\bullet}(n)(X_{\bullet})$. F_n surjective implies a suitable $x \in X_n$ exists. Hence $H_{n+1}(i)$ is surjective. □

Corollary 1.7.2. *'Aspherical' implies 'Kan'.*

Proof. Since F_n is epic for all n, so is $H_{n+1}(i)$ and hence $K_{n+1}(i)$ is. □

Some terminological loose ends: Given $X_{\bullet} \to X$, $\Delta^{\bullet}(1)(X_{\bullet}) = X_0 \times_X X_0$. Otherwise set $\Delta^{\bullet}(1)(X_{\bullet}) = X_0 \times X_0$. $\Lambda^i(1)(X_{\bullet}) = X_0$ for $i = 0, 1$. Thus H_1 and K_1 are always epic. $X_{\bullet}$ being aspherical implies $p: X_0 \to X$ is epic.

Any simplicial object without a specified augmentation may be regarded as augmented over 1.

1.8. Split simplicial objects, DEC, *and* $(-)^{\mathrm{op}}$

Definition. A simplicial object $X_{\bullet}$ is *split* if there is a family of maps $\{s_{n+1}: X_n \to X_{n+1}\}_{n \geq 0}$ (called a *contraction* for $X_{\bullet}$) satisfying all the simplicial identities involving degeneracies. A contraction for an augmented complex $X_{\bullet} \to X$ includes a map $s_0: X \to X_0$ such that $ps_0 = 1$.

Given $X_{\bullet}$ one can form a split augmented complex denoted $\mathrm{dec}(X_{\bullet})$ where $\mathrm{dec}(X_{\bullet})_n = X_{n+1}$ and where the face and degeneracy maps are those of $X_{\bullet}$ except that $d_n: X_n \to X_{n-1}$ is omitted for each n. This construction is a functor to the category of split augmented simplicial objects and contraction-preserving maps whose left adjoint is the functor which 'forgets' (omits) the augmentation and contraction. The composite functor which deletes X_0 and the maps d_n and s_n coming from each X_n and which shifts all dimensions down by one is denoted DEC$(-)$. The co-unit of the adjunction, $\mathrm{DEC}(X_{\bullet}) \to X_{\bullet}$ is, at dimension n, the face map $d_{n+1}: X_{n+1} \to X_n$.

Lemma 1.8.1. *If $X_{\bullet} = \mathrm{COSK}^n(X_{\bullet})$, then $\mathrm{DEC}(X_{\bullet}) = \mathrm{COSK}^n(\mathrm{DEC}(X_{\bullet}))$.*

Proof. For every $m \geq n+1$,

$$\Delta^{\bullet}(m)(\mathrm{DEC}(X_{\bullet})) \cong \Lambda^{m+1}(m+1)(X_{\bullet}) \cong X_{m+2} = \mathrm{DEC}(X_{\bullet})_{m+1}. \qquad \square$$

Lemma 1.8.2. *Let $X_\bullet$ be an augmented aspherical simplicial set. Then $X_\bullet \to X$ is split.*

Proof. Begin by choosing any section $s_0 : X \to X_0$. Assume inductively that a suitable $s_n : X_{n-1} \to X_n$ has been defined. Let $q_i : X_n \to \Delta^\bullet(n+1)(X_\bullet)$ be the i-th degeneracy for the simplicial kernel and define $q_{n+1} : X_n \to \Delta^\bullet(n+1)(X_\bullet)$ by

$$q_{n+1}x = (s_n d_0 x, s_n d_1 x, \dots, s_n d_n x, x).$$

Now choose a splitting $s' : \Delta^\bullet(n+1)(X_\bullet) \to X_{n+1}$ for the surjection $F : X_{n+1} \to \Delta^\bullet(n+1)(X_\bullet)$ such that $s_i = s' q_i$ for each $i = 0, \dots, n$. Then define $s_{n+1} : X_n \to X_{n+1}$ by $s_{n+1} = s' q_{n+1}$. This satisfies all the applicable identities since $q_{n+1} = F s' q_{n+1} = F s_{n+1}$. $\square$

Given $X_\bullet$ one may define another simplicial object $(X_\bullet)^{\mathrm{op}}$ by reversing the numbering of the face and degeneracy maps, e.g. $d_i^{\mathrm{op}} : X_n^{\mathrm{op}} \to X_{n-1}^{\mathrm{op}}$ is d_{n-i}.

Later we will need the simplicial object $\mathrm{DEC}(X_\bullet^{\mathrm{op}})^{\mathrm{op}}$ which we will denote $\mathrm{DEC}^{\mathrm{op}}(X_\bullet)$. The functor $\mathrm{DEC}^{\mathrm{op}}(-)$ is like DEC except that it 'forgets' the low numbered face and degeneracy maps at each dimension.

1.9. Exact fibrations

Definition. The map $f_\bullet : X_\bullet \to Y_\bullet$ is an *exact fibration at dimension n* if the square

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ \Lambda^i(n)(X_\bullet) & \longrightarrow & \Lambda^i(n)(Y_\bullet) \end{array}$$

is a pullback for each $i = 0, \dots, n$. It is an *exact fibration* if this condition holds for all n.

One may visualize this concept as follows: if the image in $Y_\bullet$ of an open i-horn in $X_\bullet$ is filled by $y \in Y_n$ then there's a unique $x \in X_n$ which fills the open i-horn in $X_\bullet$ and such that $f_n x = y$. An element of X_n is thus

$$((x_0, \dots, -, \dots, x_n), y) \in \Lambda^i(n)(X_\bullet) \times Y_n$$

such that $d_j y = x_j$ for all $j \neq i$.

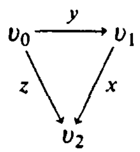
Lemma 1.9.1. *Suppose $f_\bullet : X_\bullet \to Y_\bullet$ is an exact fibration. Then:*

(i) *$Y_\bullet$ is a Kan complex implies $X_\bullet$ is a Kan complex.*
(ii) *$Y_\bullet$ aspherical implies $X_\bullet$ aspherical.*
(iii) *$Y_\bullet \cong \mathrm{COSK}^n(Y_\bullet)$ implies $X_\bullet \cong \mathrm{COSK}^n(X_\bullet)$.* $\square$

2. Groupoids, groupoid actions and torsors

2.1. Groupoids

Recall that a groupoid G is a partial binary operation on a given set where $x(yz)=(xy)z$ if either side is defined, and where each element has unique left and right units and a unique inverse. One may visualize elements of G as directed edges with specified vertices. Then the equation $xy=z$ corresponds to the picture:

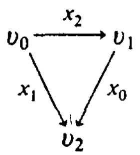


The property that any element of this equation is uniquely determined by the other two suggests the following reformulation.

Definition. A *groupoid* in $\mathscr{C}$ is a simplicial object $G_\bullet$ satisfying the axiom
GPD: For all $m>1$ and each $i=0,\dots,m$, the map $G_m \to \Lambda^i(m)(G_\bullet)$ is an isomorphism.

Let us examine this definition in some detail. The isomorphisms $G_2 \to \Lambda^i(2)(G_\bullet)$ imply that G_2 is a subobject of $\Delta^\bullet(2)(G_\bullet)$. An element of G_2 is thus (x_0,x_1,x_2) where any two of the components uniquely determine the third.

The picture is



and one traditionally writes $x_1=x_0x_2$. There is no reason to single out x_1 for special attention however.

The axiom GPD also implies that $G_\bullet \cong \mathrm{COSK}^2(G_\bullet)$ since for any $m \geq 3$ an element of $\Lambda^i(m)(G_\bullet)$ is a matrix whose i-th row is missing; the missing row is uniquely determined by the given ones and must be in G_{m-1} by GPD. An element of G_3 is thus a matrix whose rows are in G_2:

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1' & x_2' \\ x_1 & x_1' & x_2'' \\ x_2 & x_2' & x_2'' \end{bmatrix}$$

The simplicial identity $d_1d_2=d_1d_1$ applied to this matrix yields $x_0x_2'=x_0(x_2x_2'')=x_1x_2''=(x_0x_2)x_2''$, i.e. associativity. Given $x \in G_1$, then $s_0x=(x,x,s_0d_1x)$ and

$s_1x=(s_0d_0x,x,x)$. Thus $x(s_0d_1x)=x=(s_0d_0x)x$, yielding right and left units for x. Denote these ${}_x1$ and 1_x respectively. One may find 'x^{-1}' using $(x, 1_x, -)\in\Lambda^2(2)(G_\bullet)$ and then constructing an appropriate matrix to show that $(x, 1_x, x^{-1})$ *and* $(x^{-1}, {}_x1, x)$ are in G_2.

The class of groupoids and simplicial maps between them forms a category Gpd($\mathscr{C}$). The category of group objects of $\mathscr{C}$ is the full subcategory of GPD($\mathscr{C}$) consisting of those $G_\bullet$ with $G_0=1$.

2.2. Groupoid actions

Recall that the action of a group G on a set E is a map $E\times G\to E$ which sends $(y,x)\in E\times G$ to an element of E denoted yx and which satisfies the equations $y(xx')=(yx)x'$ and $y1=y$.

This concept has a simplicial description. Consider the simplicial object $E_\bullet$ where $E_0=E$, $E_1=E\times G$ and $E_2=E\times G^2$. The face maps $d_0,d_1:E_1\to E_0$ are $d_0(y,x)=y$ and $d_1(y,x)=yx$. The map $s_0:E_0\to E_1$ is $s_0(y)=(y,1)$. The face maps $d_0,d_1,d_2:E_2\to E_1$ are $d_0(y,x,x')=(y,x)$, $d_1(y,x,x')=(y,xx')$ and $d_2(y,x,x')=(yx,x')$. In fact $E_\bullet$ is a groupoid where $(y,x)(yx,x')=(y,xx')$ and $(y,x)^{-1}=(yx,x^{-1})$.

If we regard G as a groupoid $G_\bullet$ (i.e. with $G_0=1$ and $G_1=G$) then there is a groupoid map $\alpha_\bullet:E_\bullet\to G_\bullet$ defined by $\alpha_1(y,x)=x$. It is easy to check that an arbitrary $\alpha_\bullet:E_\bullet\to G_\bullet$ corresponds to an action of $G_\bullet$ on E_0 iff $\alpha_\bullet$ is an exact fibration in dimensions ≥ 1.

Definition. A *groupoid action* of the groupoid $G_\bullet$ is a simplicial map $\alpha_\bullet:E_\bullet\to G_\bullet$ which is an exact fibration in dimensions ≥ 1.

It follows immediatly from this definition that $E_\bullet$ must be a groupoid. Given a groupoid action $\alpha_\bullet$ then $\alpha_0d_0=d_0\alpha_1$ is a pullback and, since $G_2=\Lambda^i(2)(G_\bullet)$, then also $E_2=\Lambda^i(2)(E_\bullet)$. An element of E_1 is (y,x) where $\alpha_0y=d_0x$, and an element of E_2 is a matrix

$$\begin{bmatrix} y_0 & y_1 & y_2 \\ x_0 & x_1 & x_2 \end{bmatrix}$$

where $(x_0,x_1,x_2)\in G_2$ and $(y_i,x_i)\in E_1$. Denote $d_1(y,x)$ by yx. The simplicial identity $d_1d_2=d_1d_1$ applied to this element of E_2 yields $(y_0x_0)x_2=y_0(x_0x_2)$ since $y_1=y_0$, $y_2=y_0x_0$, and $x_1=x_0x_2$. Similarly, $y(s_0\alpha_0y)=y$ where $s_0\alpha_0y$ is a unit of the groupoid G.

Definition. A groupoid action $\alpha_\bullet:E_\bullet\to G_\bullet$ is *principal* if $E_1\rightrightarrows E_0$ is an equivalence pair.

Assuming $\alpha_\bullet$ is a principal groupoid action in an exact category, then $E_1\rightrightarrows E_0$ has a coequalizer $p:E_0\to X$ of which it is the kernel pair.

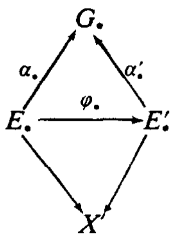
Definition. A groupoid action $\alpha_\bullet : E_\bullet \to G_\bullet$ is a *1-dimensional torsor over X* (or simply '*1-torsor*') if $E_\bullet$ is augmented over $X, E_\bullet \cong \mathrm{COSK}^0(E_\bullet)$ and $E_\bullet$ is aspherical.

Remarks. The condition $E_\bullet \cong \mathrm{COSK}^0(E_\bullet)$ implies $E_1 \rightrightarrows E_0$ is the kernel pair of $E_0 \to X$. 'Asphericity' implies $E_0 \to X$ is the coequalizer of d_0, d_1.

Lemma 2.2.1. *Given $E_\bullet \to X$ where $E_\bullet \cong \mathrm{COSK}^0(E_\bullet)$ and given a simplicial map $\alpha_\bullet : E_\bullet \to G_\bullet$ where $G_\bullet$ is a groupoid, then $\alpha_\bullet$ is a 1-torsor iff $E_0 \to X$ is epi and $\alpha_0 d_0 = d_0 \alpha_1$ is a pullback.*

Proof. $E_\bullet \cong \mathrm{COSK}^0(E_\bullet)$ implies $\alpha_\bullet$ is an exact fibration in dimensions >1. The pullback assumption makes $\alpha_\bullet$ a groupoid action. $E_0 \to X$ being epi makes $E_\bullet$ aspherical. □

Definition. A *map of 1-torsors* under $G_\bullet$ over X is a simplicial map $\varphi_\bullet : E_\bullet \to E'_\bullet$ such that

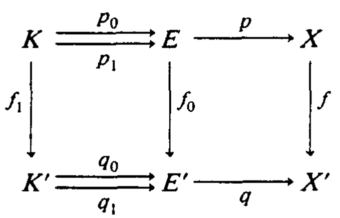


Denote the category of 1-torsors over X under G by TORS(X; $G_\bullet$).

2.3. Basic facts concerning torsors

We begin with an important lemma due to Grothendieck [10, Proposition 4.2].

Lemma 2.3.1. *In the diagram below, suppose p is an epi, p_0 and p_1 the kernel pair of p, q_0 and q_1 the kernel pair of q, $f_0 p_i = q_i f_1$ for $i = 0, 1$ and $fp = qf_0$. Then if $f_0 p_0 = q_0 f_1$ is a pullback, so is $qf_0 = fp$.*



Proof. Apply the Embedding Theorem. Suppose E'' is the pullback of

$$X \xrightarrow{f} X' \xleftarrow{q} E'.$$

We will find the inverse of the unique map $E \to E''$ defined by $y \mapsto (py, f_0 y)$. Write $y_0 \sim y_1$ iff $(y_0, y_1) \in K$ iff $py_0 = py_1$. Similarly $y_0' \sim y_1'$ for elements of E'. (These are equivalence relations.) Let $s: X \to E$ be a section for the surjection p. Since $f_0 p_0 = q_0 f_1$ is a pullback, $y' \sim f_0 y$ implies there exists a unique $y_1 \in E$ such that $y_1 \sim y$ and $y' = f_0 y_1$. If $(x, y') \in E''$ (so that $fx = qy'$) then $f_0 sx \sim y'$ since $qf_0 sx = fpsx = fx = qy'$. Thus there is a unique $y_1 \in E$ such that $y_1 \sim sx$ (equivalently $py_1 = x$) and $fy_1 = y'$. Then define $E'' \to E$ by (x, y') $\quad y_1$. □

Proposition 2.3.2. TORS(X; $G_\bullet$) *is a groupoid.*

Proof. A map $\varphi_\bullet : E_\bullet \to E_\bullet'$ of 1-torsors includes the diagram

$$\begin{array}{ccccc}
E_1 & \xrightarrow{\varphi_1} & E_1' & \xrightarrow{\alpha_1} & G_1 \\
{\scriptstyle d_0}\downarrow\downarrow{\scriptstyle d_1} & & {\scriptstyle d_0}\downarrow\downarrow{\scriptstyle d_1} & & \downarrow\downarrow \\
E_0 & \xrightarrow{\varphi_0} & E_0' & \xrightarrow{\alpha_0} & G_0 \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \\
X & \xrightarrow{1} & X & &
\end{array}$$

Then $\varphi_0 d_0 = d_0 \varphi_1$ is a pullback and Grothendieck's lemma implies $p'\varphi_0 = 1_X p$ is a pullback. Hence φ_0 is an isomorphism. Similarly, φ_m is an isomorphism for all $m \geq 1$. □

Given $f: X' \to X$, any augmented simplicial object $E_\bullet \to X$ may be 'pulled back along f' to yield a simplicial object $E_\bullet' \to X'$ where

$$\begin{array}{ccc}
E_m' & \longrightarrow & E_m \\
\downarrow & & \downarrow{\scriptstyle pd_0^m} \\
X' & \longrightarrow & X
\end{array}$$

is a pullback. An element of E_m' is $(x', y) \in X' \times E_m$ such that $fx' = pd_0^m y$. In that case, $d_i(x', y) = (x', d_i y)$.

Proposition 2.3.3. *Pulling back along $f: X' \to X$ induces a functor*

$$\mathrm{TORS}(f;\ G_\bullet): \mathrm{TORS}(X;\ G_\bullet) \to \mathrm{TORS}(X';\ G_\bullet).$$

Proof. Pullbacks preserve simplicial kernels and epimorphisms, and the composite of pullback squares is a pullback square. Apply Lemma 2.2.1. □

Remark. By Grothendieck's lemma, every map of 1-torsors arises from a pullback.

Definition. The 1-torsor $E_\bullet \to G_\bullet$ is *split* if $E_\bullet$ is split as a simplicial object.

Proposition 2.3.4. *For any groupoid $G_\bullet$, $\mathrm{DEC}(G_\bullet) \to G_\bullet$ is a torsor over G_0.*

Proof. $\mathrm{DEC}(G_\bullet)$ is augmented over G_0 and split. Now

$$\Delta^{\bullet}(1)(\mathrm{DEC}(G_\bullet)) \cong \Lambda^2(2)(G_\bullet) \cong G_2 \cong \mathrm{DEC}(G_\bullet)_1;$$

similarly, $\mathrm{DEC}(G_\bullet)_m \cong \Delta^{\bullet}(m)(\mathrm{DEC}(G_\bullet))$. Hence $\mathrm{DEC}(G_\bullet) \cong \mathrm{COSK}^0\mathrm{DEC}(G_\bullet))$. Also, the pullback of

$$G_1 \xrightarrow[d_0]{} G_0 \xleftarrow[d_0]{} G_1$$

is

$$\Lambda^1(2)G_\bullet) \cong G_2 \cong \mathrm{DEC}(G_\bullet)_1.$$

Apply Lemma 2.2.1. □

Remark. As a torsor under $G_\bullet$, $\mathrm{DEC}(G_\bullet)$ is just the action of $G_\bullet$ on itself by right translation. It is a split torsor.

Lemma 2.3.5. *If $\alpha_\bullet : E_\bullet \to G_\bullet$ is a split torsor, then $\alpha_\bullet$ factors through* $\mathrm{DEC}(G_\bullet) \to G_\bullet$.

Proof. Define $f_\bullet : E_\bullet \to \mathrm{DEC}(G_\bullet)$ by $f: X \to G_0$, $f = \alpha_0 s_0$ and $f_n = \alpha_{n+1} s_{n+1}$. (Recall that $s_{n+1} : G_{n+1} \to G_{n+2}$ is part of the contraction for $\mathrm{DEC}(G_\bullet)$.) □

Remark. The pullback of the torsor $\mathrm{DEC}(G_\bullet) \to G_\bullet$ along any $X \to G_0$ is a split torsor over X. Hence the split torsors in $\mathrm{TORS}(X;\, G_\bullet)$ are the elements of the groupoid $\mathscr{C}(X, G_\bullet)$.

2.4. Extension of the structural groupoid

The goal of this section is to prove that a groupoid map $\varphi_\bullet : G_\bullet \to G'_\bullet$ induces a functor (up to isomorphism)

$$\mathrm{TORS}(X;\, \varphi_\bullet) : \mathrm{TORS}(X;\, G_\bullet) \to \mathrm{TORS}(X;\, G'_\bullet).$$

To motivate the construction, suppose $\varphi : G \to G'$ is a homomorphism of groups and that G acts principally on E with coequalizer $p : E \to X$. Then G acts principally on $E \times G'$ by $(y, x')x = (yx, \varphi(x)^{-1}x')$. Denote the set of orbits of this action by E', and denote the orbit (y, x') represents by $[y, x']$. Then there's an action of G' on E' defined by $[y, x']x'' = [y, x'x'']$ which is principal and has orbit set X. Here, the map $E' \to X$ sends $[y, x']$ to the orbit py. E' is a torsor under G' over X.

This construction appears classically in the construction of a coordinate bundle from a system of coordinate transformations [18]. Another example occurs in a diagram of short exact sequences in an abelian category:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & (*) & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

where $(*)$ is a pushout. A acts on E over X, and φ induces an action of A' on E' over X corresponding to the bottom short exact sequence.

We will rephrase the classical extension of the structural group construction in simplicial terms so that it will apply to groupoid actions in an exact category $\mathscr{C}$.

Theorem 2.4.1. *Let* $\varphi_\bullet : G_\bullet \to G'_\bullet$ *and* $\psi_\bullet : G'_\bullet \to G''_\bullet$ *be groupoid maps. Then:*

(A) *There is a functor* $\mathrm{TORS}(X;\, \varphi_\bullet) : \mathrm{TORS}(X;\, G_\bullet) \to \mathrm{TORS}(X;\, G'_\bullet)$

(B) $\mathrm{TORS}(X;\, \psi_\bullet\varphi_\bullet) \cong \mathrm{TORS}(X;\, \psi_\bullet)\mathrm{TORS}(X;\, \varphi_\bullet)$. (*That is,* $\mathrm{TORS}(X;\, -)$ *is a functor up to isomorphism.*)

Proof. (A) Consider diagram (1) in $\mathrm{Simpl}(\mathscr{C})$.

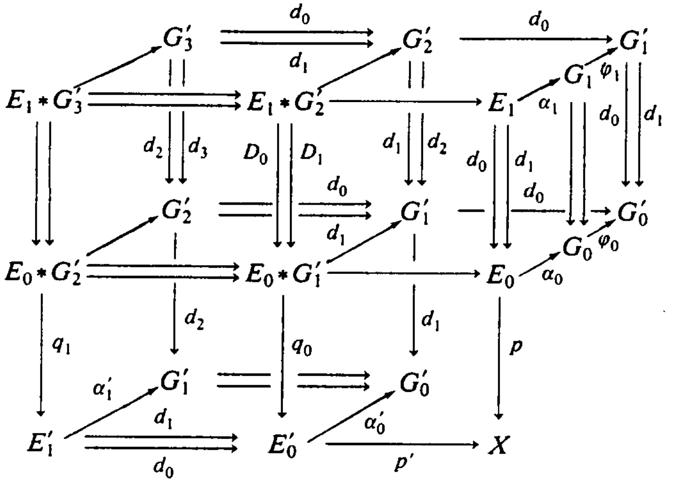
$$\begin{array}{ccccccc} \cdots (\mathrm{DEC}^{\mathrm{op}})^2(G'_\bullet) & \rightrightarrows & \mathrm{DEC}^{\mathrm{op}}(G'_\bullet) & \longrightarrow & G'_\bullet & \overset{\varphi_\bullet}{\longleftarrow} & \\ \uparrow & & \uparrow & & \uparrow & & G_\bullet \\ \cdots \quad E_\bullet * G'_{\bullet+2} & \rightrightarrows & E_\bullet * G'_{\bullet+1} & \longrightarrow & E_\bullet & \underset{\alpha_\bullet}{\longleftarrow} & \end{array} \tag{1}$$

The top row is itself a simplicial object which, written fully, is diagram (2).

$$\begin{array}{ccccccc} G'_5 & \Rrightarrow & G'_4 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & G'_3 & \overset{d_0}{\longrightarrow} & G'_2 \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\ G'_4 & \Rrightarrow & G'_3 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & G'_2 & \overset{d_0}{\longrightarrow} & G'_1 \\ \downdownarrows & & {\scriptstyle d_2}\downdownarrows{\scriptstyle d_3} & & {\scriptstyle d_1}\downdownarrows{\scriptstyle d_2} & & \downdownarrows \\ G'_3 & \Rrightarrow & G'_2 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & G'_1 & \overset{d_0}{\longrightarrow} & G'_0 \\ \downarrow & & {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} & & \\ G'_2 & \Rrightarrow & G'_1 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & G'_0 & & \end{array} \tag{2}$$

The n-th *row* of (2) is $\mathrm{DEC}^n(G'_\bullet)$ and (2) commutes simplicially. The bottom row of (1) is the simplicial object (in Simpl($\mathscr{C}$)) obtained by pulling back along $\varphi_{\bullet}\alpha_{\bullet}: E_{\bullet} \to G_{\bullet} \to G'_{\bullet}$. The object $E_{\bullet} * G'_{\bullet+n}$ is augmented over what we will denote E'_n (a coequalizer). This results in a simplicial $E'_{\bullet}$ augmented over X together with an induced $\alpha'_{\bullet}: E'_{\bullet} \to G'_{\bullet}$. We will show that $\alpha'_{\bullet}$ is a torsor.

Diagram (1), when drawn fully, is a rectangular lattice of objects and maps. That part of it which corresponds to the bottom will be called the 'front plane'; the top row will be called the 'rear plane'. The key portion of this lattice is shown in diagram (3).



(3)

Every *row* of the front plane, $E_k * G'_{\bullet+k}$, is the pullback along $\varphi_n\alpha_n: E_n \to G_n \to G'_n$ of $\mathrm{DEC}^{n+1}(G'_{\bullet})$, and is, in fact, a torsor under $G'_{\bullet}$ since $\mathrm{DEC}^{n+1}(G'_{\bullet})$ is.

It is easy to verify that $D_0, D_1: E_1 * G'_2 \rightrightarrows E_0 * G'_1$ is an equivalence pair since $d_1, d_2: G'_2 \rightrightarrows G'_1$ and $d_0, d_1: E_1 \rightrightarrows E_0$ are. Similarly, $E_1 * G'_n \rightrightarrows E_0 * G'_{n-1}$ is an equivalence pair for all $n > 1$.

Let us write down D_0 and D_1 explicitly, observing that $E_1 * G'_2$ is the pullback of

$$E_1 \xrightarrow[\varphi_1\alpha_1]{} G'_1 \xleftarrow[d_0]{} G'_2$$

and that $E_0 * G'_1$ is the pullback of

$$E_0 \xrightarrow[\varphi_0\alpha_0]{} G'_0 \xleftarrow[d_0]{} G'_1.$$

We have $((y,x),(x'_0,x'_1,x'_2)) \in E_1 * G'_2$ where $(y,x) \in E_1$, $(x'_0,x'_1,x'_2) \in G'_2$ and $\varphi_1\alpha_1(y,x) = \varphi_1(x) = d_0(x'_0,x'_1,x'_2) = x'_0$. Eliminating redundant elements (e.g. $x'_0 = \varphi_1 x$ and $x'_2 = x_0'^{-1}x'_1$), an element of $E_1 * G'_2$ is $(y,x,x') \in E_0 \times G_1 \times G'_1$ such that $(y,x) \in E_1$ and $(\varphi_1 x, x', -) \in \Lambda^2(2)(G'_{\bullet}) \cong G'_2$. The maps D_0 and D_1 are then determined componentwise as $D_0(y,x,x') = (y,x)$ and $D_1(y,x,x') = (yx, \varphi_1 x^{-1}x')$. (Compare with the introductory remarks of this section.) $E_0 * G'_1 \to E'_0$ is the coequalizer of D_0 and D_1.

We will apply Lemma 2.2.1 to show that $\alpha'_\bullet : E'_\bullet \to G'_\bullet$ is a torsor over X. That is, we must show

(i) $E'_\bullet \cong \mathrm{COSK}^0(E'_\bullet)$,

(ii) $\alpha'_0 d_0 = d_0 \alpha'_1$ is a pullback,

(iii) p' is epic.

The first two facts will follow from lemmas we will establish separately (see Lemma 2.4.4 for (i) and Lemma 2.4.5 for (ii)). As for (iii), $p'q_0$ is the composite of epis $E_0 * G'_1 \longrightarrow E_0 \xrightarrow[p]{} X$. Hence p' is epic.

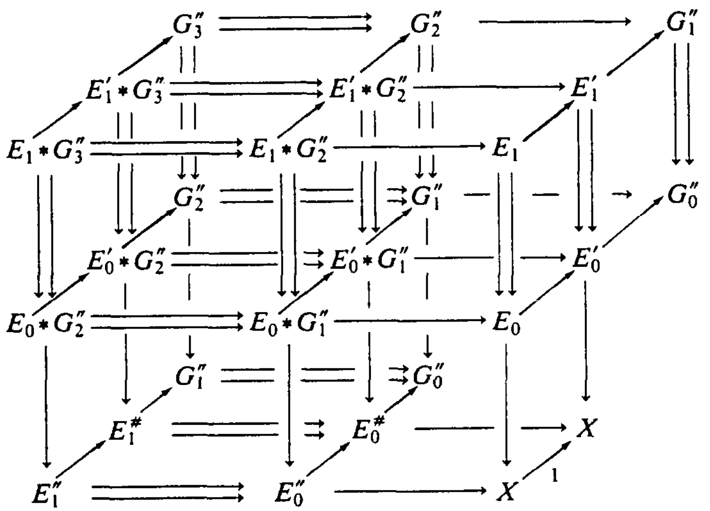
(B) Let

$$E'_\bullet \cong \mathrm{TORS}(X;\ \varphi_\bullet)(E_\bullet), \qquad E''_\bullet \cong \mathrm{TORS}(X;\ \psi_\bullet\varphi_\bullet)(E_\bullet),$$

and

$$E^{\#}_\bullet \cong \mathrm{TORS}(X;\ \psi_\bullet)(E'_\bullet).$$

Consider diagram (4).



(4)

The front plane of (4) is from the lattice diagram for $\mathrm{TORS}(X;\ \psi_\bullet\varphi_\bullet)(E_\bullet)$ and the middle plane of (4) is the front plane of the lattice for $\mathrm{TORS}(X;\ \psi_\bullet)\mathrm{TORS}(X;\ \varphi_\bullet)$. In order to obtain the map $E''_\bullet \to E^{\#}_\bullet \to G''_\bullet$ at the bottom, we will define maps $E_m * G''_{n+1} \to E'_m * G''_{n+1}$ so that each square

$$\begin{array}{ccc} E_m * G''_{n+1} & \xrightarrow[h_m * 1]{} & E'_m * G''_{n+1} \\ \Big\downarrow{\scriptstyle \langle 1, d_i \rangle} & & \Big\downarrow{\scriptstyle \langle 1, d_i \rangle} \\ E_m * G''_n & \xrightarrow[h_m * 1]{} & E'_m * G''_n \end{array}$$

is a pullback, thus showing that the horizontal plane of (4) is a $G_\bullet''$-equivariant map.

To do this we use the simplicial map $h_\bullet : E_\bullet \to E_\bullet'$ given at dimension 0 by $h_0(y) = q_0(y, s_0\varphi_0\alpha_0 y)$ with q_0 as in diagram (3).

$$E_0 \xrightarrow[\langle 1, s_0\varphi_0\alpha_0\rangle]{} E_0 * G_1' \xrightarrow[q_0]{} E_0'.$$

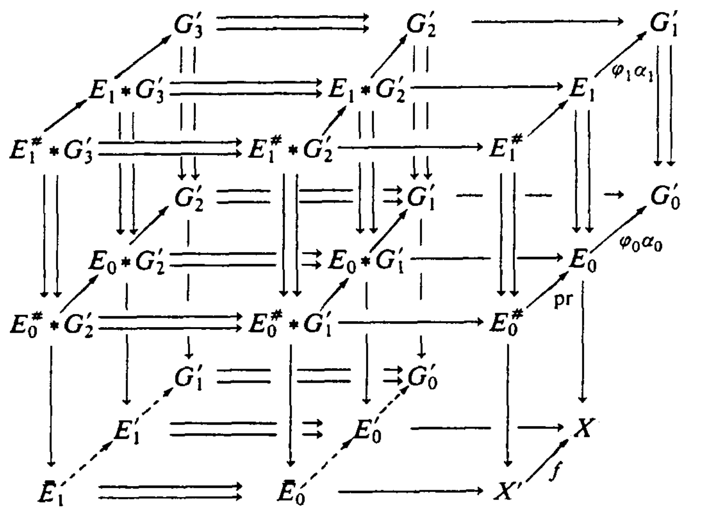
This extends uniquely to $h_\bullet$ and is equivariant: $\alpha_\bullet' h_\bullet = \varphi_\bullet \alpha_\bullet$. (See Lemma 2.4.3.) Now define $(h_m * 1)(y, x'') = (h_m y, x'')$. This induces a torsor map $E_\bullet'' \to E_\bullet^*$. □

Theorem 2.4.2 (naturality of TORS(X; –) and TORS(–; $G_\bullet$)). *Given $f : X' \to X$ and a groupoid map $\varphi_\bullet : G_\bullet \to G_\bullet'$, then the diagram*

$$\begin{array}{ccccc}
\mathrm{TORS}(X;\, G_\bullet) & \longrightarrow & \mathrm{TORS}(X;\, G_\bullet') & \quad & E_\bullet \longmapsto E_\bullet' \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{TORS}(X';\, G_\bullet) & \longrightarrow & \mathrm{TORS}(X';\, G_\bullet') & & E_\bullet^*
\end{array}$$

commutes up to isomorphism.

Proof. The commutative diagram (5) contains the proof.



(5)

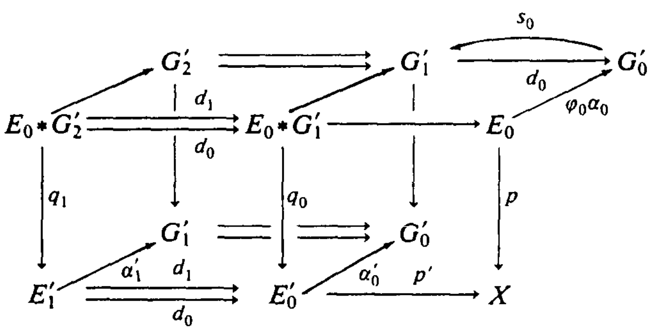
Its front plane is from the lattice diagram for TORS(X'; $\varphi_\bullet$)($E_\bullet^*$) and the middle plane is from the one for TORS(X; $\varphi_\bullet$)($E_\bullet'$). $E_\bullet^*$ is TORS^1(f; $G_\bullet$)($E_\bullet$). The maps from the front to the middle plane are uniquely determined; in fact

$$\begin{array}{ccccc}
E_0^* * G_m' & \longrightarrow & E_0 * G_m' & \longrightarrow & G_m' \\
\downarrow & & \downarrow & & \downarrow \\
E_0^* & \xrightarrow[\mathrm{pr}]{} & E_0 & \xrightarrow[\varphi_0\alpha_0]{} & G_0'
\end{array}$$

is a composite of pullbacks. The dotted map $\bar{E}_\bullet \to E'_\bullet$ is thus determined and is easily seen to be $G'_\bullet$-equivariant. By Lemma 2.3.1, $\bar{E}_\bullet \cong \mathrm{TORS}(f;\, G'_\bullet)(E'_\bullet)$. □

Lemma 2.4.3. *Let $\varphi_\bullet : G_\bullet \to G'_\bullet$ be a groupoid map. Let $\alpha_\bullet : E_\bullet \to G_\bullet$ be a torsor over X and let $\alpha'_\bullet : E'_\bullet \to G'_\bullet$ be* TORS(X; $\varphi_\bullet$)($E_\bullet$). *Then there is a $G_\bullet$-equivariant map $h_\bullet : E_\bullet \to E'_\bullet$ over X which is universal for $G_\bullet$-equivariant maps from $E_\bullet$ to torsors in* TORS(X; $G'_\bullet$).

Proof. Consider the following portion (6) of the lattice diagram for TORS(X; $\varphi_\bullet$)($E_\bullet$).



(6)

Define $h_0 : E_0 \to E'_0$ by $y \mapsto q_0(y, s_0\varphi_0\alpha_0 y)$. Clearly $py = p'h_0 y$ and $\alpha'_0 h_0 = \varphi_0\alpha_0$. The map $h_\bullet$ is $G_\bullet$-equivariant if $h_0 d_1 = d_1 h_1$ where $h_1(y,x) = (h_0 y, \varphi_1 x)$. That is, $h_0(yx) = (h_0 y)(\varphi_1 x)$. We will use the following facts.

(A) $q_0(y, x') = q_0(yx, (\varphi_1 x)^{-1} x')$ whenever yx is defined.

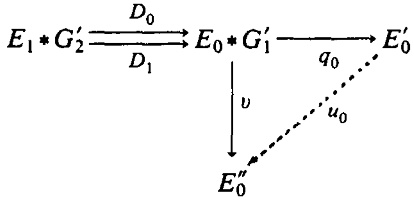
(B) The left and right units, respectively, of $\varphi_1 x$ are $s_0 d_1 \varphi_1 x = s_0\varphi_0\alpha_0(yx)$ and $s_0 d_0 \varphi_1 x = s_0 \varphi_0 \alpha_0 y$.

(C) $q_0(y, x'_0)x'_2 = q_0(y, x'_0 x'_2)$ from $q_0 d_1 = d_1 q_1$.

Now

$$h_0(yx) = q_0(yx, s_0\varphi_0\alpha_0(yx)) = q_0(y, \varphi_1 x) = q_0(y, s_0\varphi_0\alpha_0 y)\varphi_1 x = (h_0 y)(\varphi_1 x).$$

Thus $h_\bullet$ is equivariant. Now suppose $h''_\bullet : E_\bullet \to E''_\bullet$ is any $G_\bullet$-equivariant map where $E''_\bullet \in \mathrm{TORS}(X;\, G'_\bullet)$. Define $\upsilon : E_0 * G'_1 \to E''_0$ by $\upsilon(y, x') = h''_0(y)x'$. Then υ induces a unique $u_0 : E'_0 \to E''_0$ because $\upsilon D_0 = \upsilon D_1$. It is easily checked that the resulting $u_\bullet : E'_\bullet \to E''_\bullet$ is equivariant and that it satisfies $u_\bullet h_\bullet = h''_\bullet$.



□

Lemma 2.4.4. *In diagram* (7) *below, assume*

(i) $K \rightrightarrows E \to X$ *is exact*

(ii) *$K_m \rightrightarrows E_m \to X_m$ is exact for all $m \le n$*
(iii) *$K_\bullet \cong \mathrm{COSK}^n(K_\bullet)$ and $E_\bullet \cong \mathrm{COSK}^n(E_\bullet)$*
(iv)

$$\begin{array}{ccc} K_m & \xrightarrow{rd_0^m} & K \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle t_0} \\ E_m & \xrightarrow[qd_0^m]{} & E \end{array}$$

is a pullback for all m.
Then $X_\bullet \cong \mathrm{COSK}^n(X_\bullet)$ iff $K_{n+1} \rightrightarrows E_{n+1} \to X_{n+1}$ is exact.

$$\begin{array}{ccccc} K_\bullet & \rightrightarrows & E_\bullet & \longrightarrow & X_\bullet \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ K & \underset{t_1}{\overset{t_0}{\rightrightarrows}} & E & \xrightarrow{t} & X \end{array} \tag{7}$$

Proof. By Grothendieck's lemma (Lemma 2.3.1),

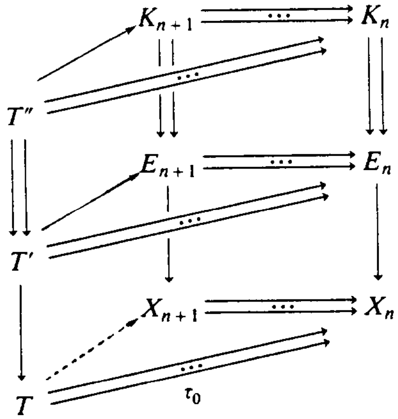
$$\begin{array}{ccc} E_m & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ E & \longrightarrow & X \end{array}$$

is a pullback for all $m \le n$. Assume first that $X_\bullet \cong \mathrm{COSK}^n(X_\bullet)$. Then in the diagram

$$\begin{array}{ccccc} K_{n+1} & \rightrightarrows & E_{n+1} & \longrightarrow & X_{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ K & \rightrightarrows & E & \longrightarrow & X \end{array}$$

the top row is the pullback of the bottom and is thus exact.

Conversely, suppose $K_{n+1} \rightrightarrows E_{n+1} \to X_{n+1}$ is exact. Let $T = \Delta^\bullet(n+1)(X_\bullet)$ and consider diagram (8).



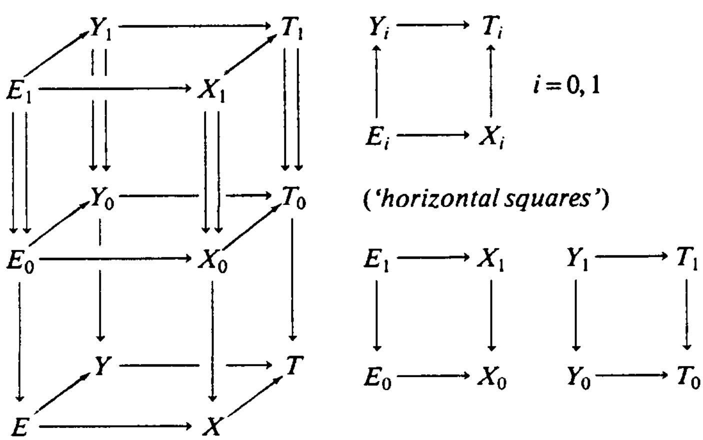
(8)

The sequence $T'' \rightrightarrows T' \to T$ is exact since it is the pullback of $K_n \rightrightarrows E_n \to X_n$ along τ_0. The maps $T'' \to K_{n+1}$ and $T' \to E_{n+1}$ exist because K_{n+1} and E_{n+1} are simplicial kernels. But since $T' \to T$ is a coequalizer, a unique $T \to X_{n+1}$ exists making the whole diagram commute. This shows that X_{n+1} is a simplicial kernel. Similarly, $X_m \cong \Delta^\bullet(m)(X_\bullet)$ for all $m > n+1$. □

Lemma 2.4.5. *In diagram* (9) *below, assume that the columns are exact and that the indicated squares are pullbacks. Then the bottom 'horizontal' square*

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & T \end{array}$$

is a pullback.



(9)

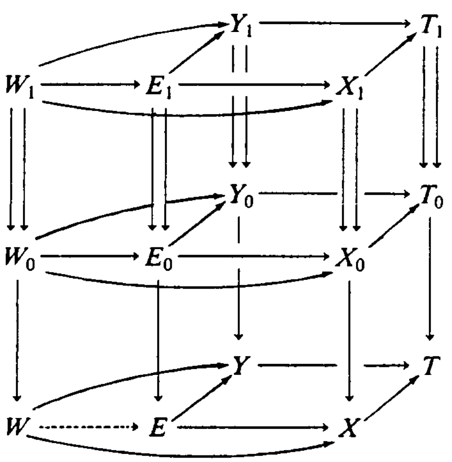
Proof. By Grothendieck's lemma (Lemma 2.3.1),

$$\begin{array}{ccc} E_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ E & \longrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_0 & \longrightarrow & T_0 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & T \end{array}$$

are pullbacks. In the following diagram (10) suppose maps $W \to X$ and $W \to Y$ are given so that

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & T \end{array}$$

commutes. Suppose further that $W_1 \rightrightarrows W_0 \to W$ is the exact sequence obtained by pulling back the 'Y-column' along $W \to Y$. Then the maps $W_i \to X_i$ are determined so that everything commutes. Since the upper two 'horizontal' squares are pullbacks, there are unique maps $W_i \to E_i$. These induce (by the exactness of the 'W-column') $W \dashrightarrow E$.



□ (10)

3. Hypergroupoids, hypergroupoid actions and torsors

An n-dimensional hypergroupoid is an algebraic structure involving a generalized composition defined simplicially. A groupoid is a 1-dimensional hypergroupoid. The discussion of hypergroupoid actions and torsors closely parallels that for the groupoid case and involves the key concept of 'attached 1-torsor'. The analog of the extension of the structural groupoid theorem will be proved in Chapter 4.

3.1. Definition and examples

Definition. An *n-dimensional hypergroupoid* ($n \geq 1$) is a simplicial object $G_\bullet$ satisfying the axiom

***n*-HYPGPD:** $G_m \to \Lambda^i(m)(G_\bullet)$ is an isomorphism for $i = 0, \ldots, m$ and all $m > n$.

A *map* of n-dimensional hypergroupoids is just a simplicial map. The category of n-dimensional hypergroupoids in the category $\mathscr{C}$ is denoted $\mathrm{Hypgpd}_n(\mathscr{C})$.

Example 1. A groupoid is a 1-dimensional hypergroupoid since GPD = 1-HYPGPD.

Example 2. If $X_\bullet = \mathrm{COSK}^{n-1}(X_\bullet)$, then $X_\bullet$ is an n-dimensional hypergroupoid. (See Section 1.6).

Example 3. Any n-dimensional hypergroupoid is also an n'-dimensional hypergroupoid for each $n' > n$ since the isomorphisms of n-HYPGPD include those of n'-HYPGPD.

Example 4. Let A be an abelian group object. Fix $n \geq 1$. Define the simplicial object $K(A, n)$ as follows. For $m = 0, \ldots, n-1$, set $K(A, n)_m = 1$. Set $K(A, n)_n = A$ and

$$K(A,n)_{n+1} = \{(a_0, \ldots, a_{n+1}) \in A^{n+2} \mid a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1} a_0 = 0\}.$$

All face and degeneracy maps below dimension $n-1$ are the identity map. The degeneracy maps $1 \to A$ are all the 'zero' element of A. At dimension n, $s_i(a) = (0, \ldots, a, a, \ldots, 0)$ where the first 'a' occurs in the i-th slot. The face maps $d_i : K(A,n)_{n+1} \to K(A,n)_n$ are $d_i(a_0, \ldots, a_{n+1}) = a_i$. In higher dimensions $K(A, n)$ consists of simplicial kernels. Thus, an $(n+2)$-simplex is a matrix whose rows are in $K(A,n)_{n+1}$. Any one of these rows is completely determined by the others; the standard double-sum argument shows that it must be in $K(A,n)_{n+1}$. If $n = 1$, then $K(A, 1)$ is simply the group object A written as a simplicial object. $K(A, n)$ is a Kan complex whose n-th homotopy group is A and all of whose other homotopy groups vanish. $K(A, n)$ is also an abelian group object in the category $\mathrm{Hypgpd}_n(\mathscr{C})$.

Example 5. Let $X_\bullet \in \mathrm{Simpl}(\mathscr{S})$ be a Kan complex. ($X_\bullet$ could be the singular complex of a topological space, for example.) There is an equivalence relation defined on X_n by: $x \sim y$ if there is a $z \in X_{n+1}$ such that $d_i z = s_{n-1} d_i x$ for $i = 0, \ldots, n-1$, $d_n z = x$ and $d_{n+1} z = y$. (This implies $d_i x = d_i y$ for all i.) Now define the simplicial object $G_\bullet$ by $G_m = X_m$ for $m = 0, \ldots, n-1$ and G_n = the equivalence classes of the equivalence relation just defined. Let $[x]$ denote the equivalence class $x \in X_n$ represents, and set $d_i([x]) = d_i x$. Now consider an element

$$(-, [x_1], \ldots, [x_{n+1}]) \in \Lambda^0(n+1)(G_\bullet).$$

Then $(-,x_1,\dots,x_{n+1})\in\Lambda^0(n+1)(X_\bullet)$. Since $X_\bullet$ is a Kan complex, there is a $y\in X_{n+1}$ such $d_iy=x_i$ for $i=1,\dots,n+1$. We then have a map $\Lambda^0(n+1)(G_\bullet)\to G_n$ sending $(-,[x_1],\dots,[x_{n+1}])$ to $[d_0y]$. This map is well defined because the class $[d_0y]$ is independent of the choices of representatives x_i and the choice of y. Now set $G_{n+1}=\Lambda^0(n+1)(G_\bullet)$, take the map just defined as d_0 and the projections to the other $[x_i]$'s as the other face maps. The result is an n-dimensional hypergroupoid called the *n-th fundamental hypergroupoid of* $X_\bullet$. The 1-dimensional version of this is the fundamental groupoid of $X_\bullet$. One may recover from the n-th fundamental hypergroupoid all the n-th homotopy groups of $X_\bullet$.

3.2. *Hyper-associativity and hyperunit laws*

There are analogs for hypergroupoids of the associativity and unit laws for groupoids. We will choose one of the $(n+1)$-ary operations (the choice being suggested by technical convenience) to illustrate these laws.

Suppose $G_\bullet$ is an n-dimensional hypergroupoid. Write $x_{n+1}=[x_0,\dots,x_n]$ iff $(x_0,\dots,x_n,x_{n+1})\in G_{n+1}$. The following matrix represents an element of G_{n+2}.

$$\begin{bmatrix} x_{00} & x_{01} & \cdots & x_{0,n+1} \\ x_{10} & x_{11} & \cdots & x_{1,n+1} \\ \vdots & \vdots & & \vdots \\ x_{n+2,0} & x_{n+2,1} & \cdots & x_{n+2,n+1} \end{bmatrix}$$

Since the i-th row is $(x_{i0},\dots,x_{in+1})$ then $x_{i,n+1}=[x_{i0},\dots,x_{in}]$. Since $x_{ji}=x_{i,j-1}$ for $i<j$ (see Section 1.6) we have

$$\begin{aligned} x_{n+2,n+1} &= [x_{n+2,0},\dots,x_{n+2,n}] = [x_{0,n+1},\dots,x_{n,n+1}] \\ &= [[x_{00},\dots,x_{0n}],[x_{10},\dots,x_{1n}],\dots,[x_{n0},\dots,x_{nn}]]. \end{aligned}$$

Also, since

$$x_{n+2,n+1}=x_{n+1,n+1}=[x_{n+1,0},\dots,x_{n+1,n}]=[x_{0n},\dots,x_{nn}],$$

we have the 'hyper-associativity law':

HA$_n$: $\quad [[x_{00},\dots,x_{0n}],[x_{10},\dots,x_{1n}],\dots,[x_{n0},\dots,x_{nn}]]=[x_{0n},x_{1n},\dots,x_{nn}].$

For example HA_2 is

$$[[x_{00},x_{01},x_{02}],[x_{00},x_{11},x_{12}],[x_{01},x_{11},x_{22}]]=[x_{02},x_{12},x_{22}].$$

The 'hyper-unit laws' correspond to the degenerate elements $s_jx\in G_{n+1}$.

HU$_{n,j}$: $\quad s_jd_{n+1}x=[s_{j-1}d_0x,s_{j-1}d_1x,\dots,x,x,\dots,s_jd_nx]$

where the x's appear in the j-th and $(j+1)$-st slots.

3.3. *Substructures of hypergroupoids*

Let $G_\bullet$ be an n-dimensional hypergroupoid. For each positive $m<n$, a certain subobject of G_{n+1} comprises the graph of an m-dimensional hypergroupoid. The groupoid ($m=1$) so determined plays a significant role in higher dimensional torsors. We'll consider the general $m\geq 1$ case first and then spell out the $m=1$ case in detail and give a few examples.

Fix m between 1 and $n-1$. Let $(*n,m)$ denote the set of conditions:

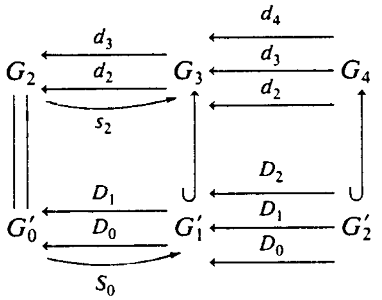
$$(*n,m) \qquad d_i x = s_{n-m-1} d_i d_{n-m} x, \quad 0\leq i\leq n-m-1,$$

where x is any simplex of dimension bigger than $n-m$. Define the simplicial object $G'_\bullet$ by setting G'_0 to be G_{n-m} and, for $k>0$

$$G'_k = \{x\in G_{n-m+k} \mid x \text{ satisfies } (*n,m)\}.$$

The face operators D_i and degeneracy operators S_i of $G'_\bullet$ are the restrictions of d_{n-m+i} and s_{n-m+i} respectively.

Example ($n=5$ and $m=3$).



Lemma 3.3.1. *$G'_\bullet$ is an m-dimensional hypergroupoid.*

Proof. An element of G'_{m+1} is $\boldsymbol{x}=(v_0,\ldots,v_{n-m-1},u_0,\ldots,u_{m+1})\in G_{n+1}$ satisfying $(*n,m)$. For $i\leq n-m-1$,

$$d_i\boldsymbol{x} = v_i = s_{n-m-1} d_i d_{n-m}\boldsymbol{x} = s_{n-m-1} d_i u_0 = s^2_{n-m-1} d_i d_{n-m} u_0.$$

$(u_0,\ldots,u_{m+1})\in \Delta^{\cdot}(m+1)(G'_\bullet)$ since for $i<j$,

$$D_i u_j = d_{n-m+i} d_{n-m+j}\boldsymbol{x} = d_{n-m+j-1} d_{n-m+i}\boldsymbol{x} = D_{j-1} u_i.$$

Furthermore, given $(u_0,\ldots,-,\ldots,u_{m+1})\in \Lambda^k(m+1)(G'_\bullet)$, then

$$(v_0,\ldots,v_{n-m-1},u_0,\ldots,-,\ldots,u_{m+1})\in \Lambda^{n-m+k}(n+1)(G_\bullet).$$

The hypergroupoid structure of $G_\bullet$ determines a unique $\bar{u}_k\in G_n$ which is easily verified to satisfy $(*n,m)$. Hence $G'_{m+1}\cong \Lambda^k(m+1)(G'_\bullet)$. Similarly $G'_q\cong \Lambda^k(q)(G'_\bullet)$ for all $q>m$ and all k. $\square$

Remark. Any map of n-dimensional hypergroupoids restricts to a map of their associated m-dimensional hypergroupoid substructures. Thus, the construction just given determines a functor $\mathrm{Hypgpd}_n(\mathscr{C}) \to \mathrm{Hypgpd}_m(\mathscr{C})$.

Example 1. Take $m=1$. Then

$$G_1' = \{x \in G_n \mid d_i x = s_{n-2} d_i d_{n-1} x \text{ for } 0 < i \leq n-2\}.$$

For $x \in G_1'$, $D_0 x = d_{n-1} x$ and $D_1 x = d_n x$. HA_n for $G_\bullet$ 'collapses' to HA_1 for $G_\bullet'$ (and similarly for $\mathrm{HU}_{i,j}$).

Example 2. Let $G_\bullet = K(A,n)$ and consider the associated groupoid. We have $(a_0, a_1, a_2) \in G_2'$ iff $(0,0,\dots,0,a_0,a_1,a_2) \in K(A,n)_{n+1}$ iff $a_2 - a_2 + a_0 = 0$. That is, the associated groupoid is $K(A,1)$, the group A itself.

Example 3. Let $X_\bullet \in \mathrm{Simpl}(\mathscr{S})$ be a Kan complex. Choose a base point $* \in X_0$, fix $n \geq 1$, and consider the subcomplex $X_\bullet'' \subseteq X_\bullet$ where $X_k'' = X_k$ for $k < n$ and $X_k'' = \{x \in X_k \mid d_0^k x = *\}$ for $k \geq n$. The n-th homotopy hypergroupoid of $X_\bullet''$ is $K(\Pi_n(X_\bullet, *), n)$ whose associated groupoid is $\Pi_n(X_\bullet; *)$. Thus the singular n-simplices of a topological space have an algebraic structure (the n-th homotopy hypergroupoid) encompassing all the n-th homotopy groups of the space.

3.4. The hypergroupoid/groupoid identities

Lemma 3.4.1. *Let $G_\bullet$ be an n-dimensional hypergroupoid and let $G_\bullet'$ be its associated groupoid. Fix $i, 0 \leq i \leq n-1$. Suppose $x_0, \dots, x_i, \dots, x_{n-1}$ and x_i' are elements of $G_1' \hookrightarrow G_n$ and suppose $x_i x_i'$ is defined in G_1'. Then*

$$[x_0, \dots, x_i x_i', \dots, x_n] = [x_0, \dots, x_i', \dots, x_{n-1}, [1_{x_0}, \dots, x_i, \dots, 1_{x_{n-1}}, x_n]].$$

Proof. Recall that for $x \in G_1'$, $1_x = S_0 D_0 x = s_{n-1} d_{n-1} x$ and that $1_x x = x$ in G_1'. If $[x_0, \dots, x_i x_i', \dots, x_n] = y$ is defined, then $[x_0, \dots, x_i', \dots, z] = y$ for some z determined, as follows, by the hypergroupoid structure. Consider the matrix in G_{n+2} defined by setting $R_j = j$-th row $= s_n x_j$ for $0 \leq j \leq n-1$ and $j \neq i$, setting

$$R_i = (*, \dots, *, x_i, x_i x_i', x_i') \in G_2' \hookrightarrow G_{n+1},$$

and setting $R_{n+1} = (x_0, \dots, x_i x_i', \dots x_n, y)$. ('$*$' denotes various degenerate elements). R_n and R_{n+2} are then uniquely determined; $R_n = (1_{x_0}, \dots, x_i, \dots, 1_{x_{n-1}}, x_n, z)$ defines z, and $R_{n+2} = (x_0, x_1, \dots, x_i', \dots, x_{n-1}, z, y) \in G_{n+1}$. R_{n+1} and R_{n+2} together yield the conclusion of the lemma. □

Corollary 3.4.2. *Suppose $x_0 x_0', x_1 x_1', \dots, x_{n-1} x_{n-1}'$ are all defined in the associated groupoid $G_\bullet'$ of the n-dimensional hypergroupoid $G_\bullet$ and suppose $[x_0 x_0', \dots, x_{n-1} x_{n-1}', x_n]$ is defined. Then*

$$[x_0', \dots, x_{n-1}', [x_0, \dots, x_{n-1}, x_n]] = [x_0 x_0', \dots, x_{n-1} x_{n-1}', x_n].$$

Proof. Apply Lemma 3.4.1 repeatedly to obtain the equalities:

$$\begin{aligned}
[x_0x_0', \ldots, x_{n-1}x_{n-1}', x_n] &= [x_0', x_1x_1', \ldots, x_{n-1}x_{n-1}', [x_0, 1, \ldots, 1, B_0]] \\
&= [x_0', x_1', x_2x_2', \ldots, x_{n-1}x_{n-1}', [1, x_1, 1, \ldots, 1, B_1]] \\
&\ \vdots \\
&= [x_0', x_1', \ldots, x_{n-1}', [1, 1, \ldots, 1, x_{n-1}, B_{n+1}]]
\end{aligned}$$

where $B_0 = x_n$, $B_{k+1} = [1, 1 \ldots, x_k, \ldots, 1, B_k]$. Then

$$\begin{aligned}
B_n &= [1, \ldots, x_{n-1}, [1, \ldots, 1, x_{n-2}, 1, B_{n-2}]] = [1, \ldots, x_{n-2}x_{n-1}, B_{n-2}] \\
&= [1, \ldots, 1, x_{n-2}, x_{n-1}, [1, \ldots, 1, x_{n-3}, 1, 1, B_{n-3}]] \\
&= \cdots = [x_0, x_1, \ldots, x_{n-1}, x_n]. \qquad \square
\end{aligned}$$

Example. Let $n = 3$ and $i = 1$. The matrix of the lemma is

$$\begin{bmatrix}
* & * & 1_{x_0} & x_0 & x_0 \\
* & * & x_1 & x_1x_1' & x_1' \\
* & * & 1_{x_2} & x_2 & x_2 \\
1_{x_0} & x_1 & 1_{x_2} & x_3 & z \\
x_0 & x_1x_1' & x_2 & x_3 & y \\
x_0 & x_1' & x_2 & z & y
\end{bmatrix}$$

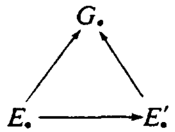
$y = [x_0, x_1', x_2, [1_{x_0}, x_1, 1_{x_2}, x_3]] = [x_0, x_1x_1', x_2, x_3]$.

3.5. *Hypergroupoid actions*

Definition. A *hypergroupoid action* of the n-dimensional hypergroupoid $G_\bullet$ is a simplicial map $\alpha_\bullet: E_\bullet \to G_\bullet$ which is an exact fibration in dimensions $\geq n$. An *equivariant map* between the hypergroupoid actions $\alpha_\bullet: E_\bullet \to G_\bullet$ and $\alpha_\bullet': E_\bullet' \to G_\bullet'$ is a commutative square

$$\begin{array}{ccc}
G_\bullet & \longrightarrow & G_\bullet' \\
\alpha_\bullet \big\uparrow & & \big\uparrow \alpha_\bullet' \\
E_\bullet & \longrightarrow & E_\bullet'
\end{array}$$

Remarks. When $n = 1$, this definition reduces to the definition of groupoid action given in Chapter 2. If $\alpha_\bullet: E_\bullet \to G_\bullet$ is a hypergroupoid action, then $E_\bullet$ itself is an n-dimensional hypergroupoid where the isomorphism $E_m \to \Lambda^i(m)(E_\bullet)$ for $m > n$ is the pullback of the corresponding isomorphism for $G_\bullet$. (See Section 2.2.) The identity map $G_\bullet \to G_\bullet$ is a hypergroupoid action. Given two actions of the hypergroupoid $G_\bullet$, a $G_\bullet$-*equivariant map* is a commutative triangle



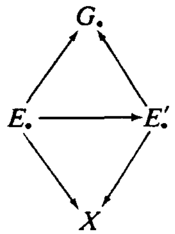
The collection of such actions of $G_\bullet$ forms a category.

3.6. *Torsors under $G_\bullet$*

Definition. Let $G_\bullet$ be an n-dimensional hypergroupoid. An action $\alpha_\bullet : E_\bullet \to G_\bullet$ is an *n-dimensional torsor over X under $G_\bullet$* if $E_\bullet$ is augmented over X, $E_\bullet \cong \mathrm{COSK}^{n-1}(E_\bullet)$ and $E_\bullet$ is aspherical. (Compare with Section 2.2).

When all the other data of this definition are clear from the context, we will speak of 'the n-torsor $E_\bullet$'.

Denote by $\mathrm{TORS}(X; G_\bullet)$ the category of torsors under $G_\bullet$ over X and 'torsor maps' under $G_\bullet$ over X where a *torsor map* is a commutative diagram:



The following lemma is convenient for checking whether a given simplicial map is a torsor. (Compare with Lemma 2.2.1.)

Lemma 3.6.1. *Let $G_\bullet$ be an n-dimensional hypergroupoid and let $\alpha_\bullet : E_\bullet \to G_\bullet$ be a simplicial map such that*

$$\begin{array}{ccc} E_n & \longrightarrow & G_n \\ \downarrow & & \downarrow \\ \Lambda^i(n)(E_\bullet) & \longrightarrow & \Lambda^i(n)(G_\bullet) \end{array}$$

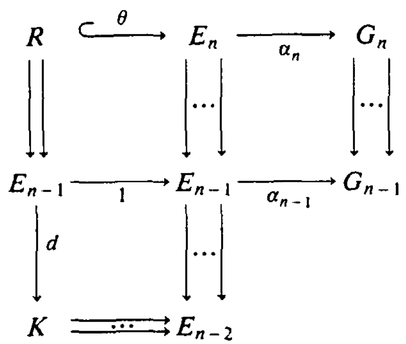
If $E_\bullet = \mathrm{COSK}^{n-1}(E_\bullet)$, then $\alpha_\bullet$ is a hypergroupoid action.

Proof. Immediate from the fact that $E_\bullet \cong \mathrm{COSK}^{n-1}(E_\bullet)$ implies $E_m \cong \Lambda^i(m)(E_\bullet)$ for $m \geq n+1$. □

Example. Let $G_\bullet$ be a n-dimensional hypergroupoid and consider $\mathrm{DEC}(G_\bullet) \to G_\bullet$. It follows immediately from n-HYPGPD that this map is a hypergroupoid action (compare with remark after Proposition 2.3.4) and using that $\mathrm{DEC}(G_\bullet) \cong \mathrm{COSK}^{n-1}(\mathrm{DEC}(G_\bullet))$. If $G_\bullet$ happened to be aspherical, e.g. if $G_\bullet$ is $K(A,n)$, then $\mathrm{DEC}(G_\bullet) \in \mathrm{TORS}(G_0; G_\bullet)$.

3.7. *The attached* 1-*torsor*

Let $G_\bullet$ be an n-dimensional hypergroupoid and suppose $\alpha_\bullet : E_\bullet \to G_\bullet$ is a torsor. Consider diagram (11).



(11)

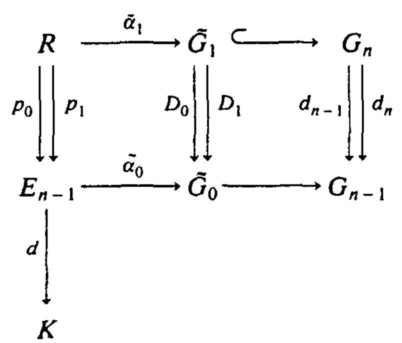
In this diagram, $K \cong \Delta^*(n-1)(E_\bullet)$, and $R \rightrightarrows E_{n-1}$ is the kernel pair of the canonical epic projection $d: E_{n-1} \to K$. The monomorphism $\theta: R \hookrightarrow E_n$ is defined by

$$\theta(y, y') = (s_{n-2}d_0y, s_{n-2}d_1y, \ldots, s_{n-2}d_{n-2}y, y, y').$$

Now $\alpha_n\theta(y,y')$ satisfies $(*n, 1)$ (see Section 3.3) since for $i = 0, \ldots, n-2$,

$$\begin{aligned} d_i\alpha_n\theta(y, y') &= \alpha_{n-1}d_i\theta(y, y') = \alpha_{n-1}s_{n-2}d_iy = s_{n-2}d_i\alpha_{n-1}y \\ &= s_{n-2}d_id_{n-1}\alpha_n\theta(y, y'). \end{aligned}$$

Let $E_{\bullet,1}$ denote $\mathrm{cosk}^1(K \leftarrow E_{n-1} \leftleftarrows R)$ and let $\tilde{G}_\bullet$ denote the associated groupoid of $G_\bullet$. (See Section 3.3.) We then have a map $\tilde{\alpha}_\bullet : E_{\bullet,1} \to \tilde{G}_\bullet$



$\tilde{\alpha}_0 = \alpha_{n-1}$ and $\tilde{\alpha}_1(y, y') = \alpha_n\theta(y, y')$. An element of $\tilde{R}$ in the pullback

$$\begin{array}{ccc} \tilde{R} & \longrightarrow & \tilde{G}_1 \\ \downarrow & & \downarrow D_0 \\ E_{n-1} & \xrightarrow{\tilde{\alpha}_0} & \tilde{G}_0 \end{array}$$

is $(y,x) \in E_{n-1} \times \tilde{G}_1$ such that $\alpha_{n-1} y = D_0 x$. If $(y,x) \in \tilde{R}$ then

$$(s_{n-2} d_0 y, \ldots, s_{n-2} d_{n-2} y, -, x) = z$$

is an element of E_n. Define $R \to \tilde{R}$ by $(y, y') \mapsto (y, \alpha_n \theta(y, y'))$ and define $\tilde{R} \to R$ by $(y,x) \mapsto (y, d_n z)$. The maps are inverses of each other. One similarly verifies that R is also the pullback of

$$E_{n-1} \xrightarrow[\tilde{\alpha}_0]{} G_0 \xleftarrow[D_1]{} G_1.$$

By Lemma 2.2.1 we have:

Lemma 3.7.1. $\tilde{\alpha}_\bullet : E_{\bullet,1} \to \tilde{G}_\bullet \in \mathrm{TORS}(K;\, G_\bullet)$. □

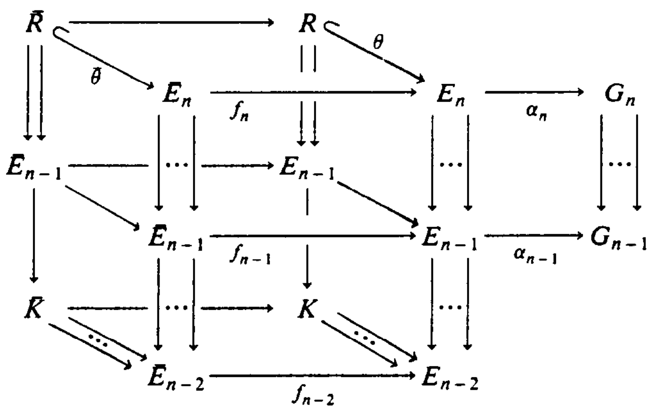
$E_{\bullet,1}$ is called the *attached* 1*-torsor* of $E_\bullet$. It is in fact the associated groupoid of $E_\bullet$ regarding $E_\bullet$ as an n-dimensional hypergroupoid.

3.8. Basic facts concerning n-torsors

The following series of propositions explain the relationship between n-torsors and their attached 1-torsors in detail. They are useful in reducing questions about n-torsors to (easier) questions about their attached 1-torsors.

Proposition 3.8.1. *Let $\alpha_\bullet : E_\bullet \to G_\bullet$ be an n-torsor over X and suppose $f_{\bullet,\mathrm{tr}} : \bar{E}_{\bullet,\mathrm{tr}} \to \mathrm{TR}^{n-2}(E_\bullet)$ is a simplicial map of the indicated $(n-2)$-truncated simplicial objects. Then $f_{\bullet,\mathrm{tr}}$ extends to a $G_\bullet$-equivariant map $f_\bullet : \bar{E}_\bullet \to E_\bullet$.*

Proof. Consider diagram (12).



(12)

In this diagram, $K = \Delta^\bullet(n-1)(E_\bullet)$, $\bar{K} = \Delta(n-1)(\bar{E}_\bullet)$ and $\bar{R} \rightrightarrows \bar{E}_{n-1} \to \bar{K}$ is the pullback of the attached 1-torsor $R \rightrightarrows E_{n-1} \to K$ along $\bar{K} \to K$. Define $\bar{E}_\bullet$ to be $\operatorname{cosk}^{n-1}(\bar{E}_{\bullet,\mathrm{tr}})$ and $f_\bullet : \bar{E}_\bullet \to E_\bullet$ to be the simplicial map thus induced. To prove that $f_\bullet$ is $G_\bullet$-equivariant we will show that $\alpha_\bullet f_\bullet : \bar{E}_\bullet \to G_\bullet$ is a hypergroupoid action. By Lemma 3.6.1 this reduces to showing that the composite square

$$\begin{array}{ccccc} \bar{E}_n & \longrightarrow & E_n & \longrightarrow & G_n \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda^i(n)(\bar{E}_\bullet) & \longrightarrow & \Lambda^i(n)(E_\bullet) & \longrightarrow & \Lambda^i(n)(G_\bullet) \end{array}$$

is a pullback for each $i = 0, \ldots, n$. Since the right hand square is already known to be a pullback it suffices to show that the left hand square is. Let $\mathbf{z} = (z_0, \ldots, z_n)$ denote an element of $\bar{K}$ where $z_i = d_i \mathbf{z}$. Then an element of $\bar{E}_{n-1}$ is $(\mathbf{z}, y) \in \bar{K} \times E_{n-1}$ such that $f_{n-2} z_i = d_i y$. An element of $\Lambda^i(n)(\bar{E}_\bullet)$ is thus

$$((z_0, y_0), \ldots, -, \ldots, (z_n, y_n))$$

where for $j < k$ and $j, k \neq i$ we have $d_j z_k = \mathbf{z}_{kj} = \mathbf{z}_{j,k-1} = d_{k-1} z_j$. It follows that

$$(y_0, \ldots, -, \ldots, y_n) \in \Lambda^i(n)(E_\bullet).$$

Let W denote the pullback of $\Lambda^i(n)(\bar{E}_\bullet) \to \Lambda^i(n)(E_\bullet) \leftarrow E_n$ (the left-hand square). An element of W is

$$((z_0, y_0), \ldots, -, \ldots, (z_n, y_n), y'_0, \ldots, -, \ldots, y'_n)$$

in $\Lambda^i(n)(\bar{E}_\bullet) \times E_n$ such that $y'_j = y_j$ for all $j \neq i$. But a unique (z_i, y_i) in $\bar{E}_{n-1}$ is thus determined $(y_i = y'_i)$ which provides a map $W \to \bar{E}_{n-1}$ and establishes that the left hand square is a pullback. □

Remark. $\bar{E}_\bullet = \operatorname{COSK}^{n-1}(\bar{E}_\bullet)$ by construction. If $\bar{E}_{\bullet,\mathrm{tr}}$ was aspherical and augmented over Y, then $\bar{E}_\bullet \in \operatorname{TORS}(Y; G_\bullet)$.

Corollary 3.8.2. *Any map of torsors $\bar{E}_\bullet \to E_\bullet \to G_\bullet$ arises from $\mathrm{TR}^{n-2}(\bar{E}_\bullet) \to \mathrm{TR}^{n-2}(E_\bullet)$ as in the previous proposition.*

Proof. Apply Grothendieck's lemma (Lemma 2.3.1) to the induced map of the attached 1-torsors. □

Corollary 3.8.3. *Any map $f : Y \to X$ induces a functor*

$$\operatorname{TORS}(f; G_\bullet) : \operatorname{TORS}(X; G_\bullet) \to \operatorname{TORS}(Y; G_\bullet).$$

Also, $\operatorname{TORS}(fg; G_\bullet) \cong \operatorname{TORS}(f; G_\bullet)\operatorname{TORS}(g; G_\bullet)$.

Proof. Given $E_\bullet \in \operatorname{TORS}(X; G_\bullet)$, form the pullback truncated simplicial object

$$\begin{array}{ccc} E_{\bullet,\mathrm{tr}} & \longrightarrow & \mathrm{TR}^{n-2}(E_\bullet) \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\ f\ } & X \end{array}$$

and apply Proposition 3.8.1. □

Proposition 3.8.4. *Let $\alpha_\bullet : E_\bullet \to G_\bullet$ be an n-torsor. Suppose $y_i \in E_{n-1}$ and $x_i \in \tilde{G}_1$ where $\tilde{G}_\bullet$ is the associated groupoid of $G_\bullet$, and assume $y_i x_i$ is defined for $i = 0, \ldots, n$. Then*

(a) $\alpha_n(y_0 x_0, \ldots, y_n x_n) = [x_0, \ldots, x_{n-1}, t]$ *where* $t = \alpha_n(y_0, \ldots, y_{n-1}, y_n x_n)$.

(b) $x_n = [v_0, \ldots, v_{n-2}, \alpha_n(y_0, \ldots, y_n), t]$ *where* $v_i = s_{n-2}\alpha_{n-1} y_i$.

Proof. Denote $(y_0, \ldots, y_n)$ by $\boldsymbol{y}$ and $(y_0 x_0, \ldots, y_n x_n)$ by $\boldsymbol{yx}$. Consider the element of E_{n+1} in (13).

$$\begin{bmatrix} * & \cdots & * & y_0 & y_0 x_0 \\ \vdots & & \vdots & \vdots & \vdots \\ * & \cdots & * & y_{n-1} & y_{n-1} x_{n-1} \\ y_0 & \cdots & y_{n-2} & y_{n-1} & y_n x_n \\ y_0 x_0 & \cdots & y_{n-2} x_{n-2} & y_{n-1} x_{n-1} & y_n x_n \end{bmatrix} \tag{13}$$

The i-th row, $0 \le i \le n-1$, is $\theta(y_i, y_i x_i) \in E_n$. The $(n+1)$-st row is $\boldsymbol{yx}$. The n-th row is then uniquely determined, as shown. Identity (a) follows from applying α_{n+1} to this matrix.

To obtain identity (b) consider the two matrices in (14).

$$\begin{bmatrix} y_0 & y_1 & \cdots & y_{n-1} & y_n x_n \\ y_0 & y_1 & \cdots & y_{n-1} & y_n \\ y_1 & y_1 & \cdots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ y_{n-1} & y_{n-1} & \cdots & * & * \\ y_n x_n & y_n & \cdots & * & * \end{bmatrix} \qquad \begin{bmatrix} * & * & \cdots & y_n & y_n x_n \\ * & * & \cdots & y_n & y_n \\ * & * & \cdots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ y_n & y_n & \cdots & * & * \\ y_n x_n & y_n & \cdots & * & * \end{bmatrix} \tag{14}$$

The i-th row, $2 \le i \le n$, of the first matrix is $s_0 y_i$. In the second matrix the 0-th row is $\theta(y_n, y_n x_n)$, the 1-st row is $s_{n-1} y_n$, the i-th row $(2 \le i \le n-1)$ is $s_{n-1} s_0 d_{n-1} y_{i-1}$, and the n-th row is $s_0 y_n$. After applying α_{n+1} to each matrix, one gets the bottom two rows of the matrix in G_{n+2} shown in (15).

$$\begin{bmatrix} v_0 & v_1 & \cdots & \alpha_n \boldsymbol{y} & t & x_n \\ * & * & \cdots & \alpha_n \boldsymbol{y} & \alpha_n \boldsymbol{y} & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ * & * & \cdots & * & * & * \\ \alpha_n \boldsymbol{y} & \alpha_n \boldsymbol{y} & \cdots & * & * & * \\ t & \alpha_n \boldsymbol{y} & \cdots & & \cdots & z \\ x_n & \alpha_n s_{n-1} y_n & \cdots & & \cdots & z \end{bmatrix} \tag{15}$$

Row 1 is $s_{n-1}\alpha_n y$, the i-th row $(2\leq i\leq n-1)$ is $s_{n-1}s_0\alpha_{n-1}y_{i-1}$ and the n-th row is $s_0\alpha_n y$. The 0-th row shows identity (b). □

Remark. This proposition relating the groupoid action of the attached 1-torsor to the hypergroupoid action $\alpha_\bullet$ will be used to prove the 'extension of the structural hypergroupoid' theorem in Chapter 4.

4. Extension of the structural hypergroupoid

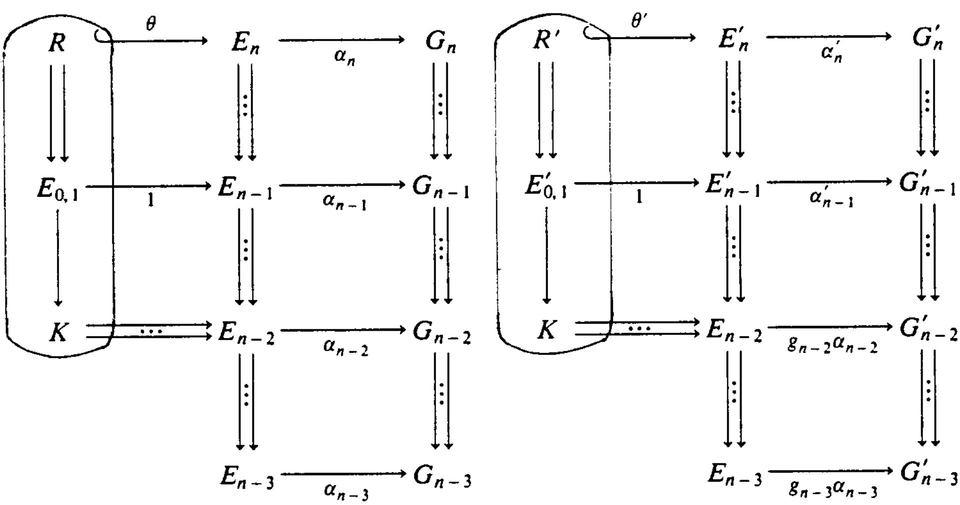
Theorem 4.1. *Let $g_\bullet: G_\bullet \to G'_\bullet$ be a map of n-dimensional hypergroupoids. Then there is a functor*

$$\mathrm{TORS}(X;\, g_\bullet): \mathrm{TORS}(X;\, G_\bullet) \to \mathrm{TORS}(X;\, G'_\bullet).$$

If $g'_\bullet: G'_\bullet \to G''_\bullet$ is another hypergroupoid map, then

$$\mathrm{TORS}(X;\, g'_\bullet g_\bullet) \cong \mathrm{TORS}(X;\, g'_\bullet)\mathrm{TORS}(X;\, g_\bullet).$$

Proof. *Outline:* Let $\alpha_\bullet: E_\bullet \to G_\bullet \in \mathrm{TORS}(X;\, G_\bullet)$. The map $g_\bullet$ induces a map $\tilde{g}_\bullet: \tilde{G}_\bullet \to \tilde{G}'_\bullet$ of the associated groupoids. Let $E_{\bullet,1}$ be the attached 1-torsor of $E_\bullet$ and $E'_{\bullet,1} = \mathrm{TORS}(K;\, \tilde{g}_\bullet)(E_{\bullet,1})$ where $K = \Delta^\bullet(n-1)(E_\bullet)$. (See Section 2.4.) This new 1-torsor will be the attached 1-torsor of an n-torsor under $G'_\bullet$. Diagram (16) summarizes the construction.



(16)

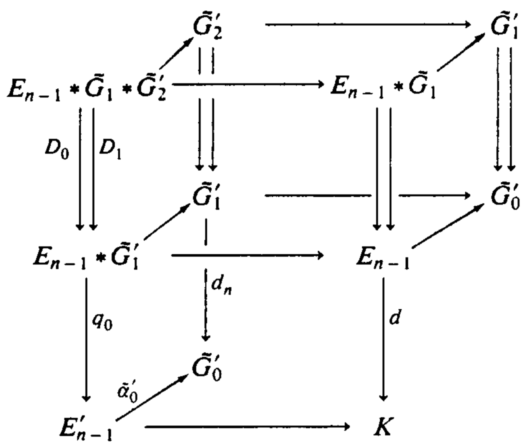
As the picture suggests, we form a new simplicial object $E'_\bullet$ from $E_\bullet$ by truncating $E_\bullet$ at dimension $n-2$, replacing E_{n-1} by $E'_{0,1}$ and setting E'_m to be the simplicial kernel for $m\geq n$. $E'_\bullet$ is aspherical and satisfies $E'_\bullet \cong \mathrm{COSK}^{n-1}(E'_\bullet)$ by construction.

We will show that a map $\alpha'_\bullet: E'_\bullet \to G'_\bullet$ exists making $E'_\bullet$ an n-torsor under $G'_\bullet$ with attached 1-torsor $E'_{\bullet,1}$.

In other words, the extension of the structural hypergroupoid reduces, at the attached 1-torsor level, to the extension of the structural groupoid.

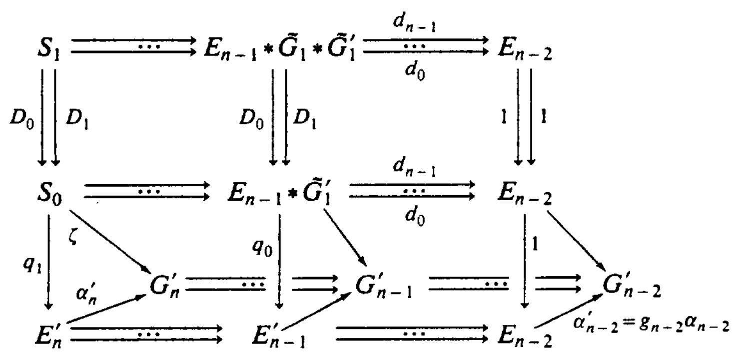
The proof that $E'_\bullet$ is an n-torsor is divided into two parts: (I) the definition of $\alpha'_\bullet$; (II) the verification that $\alpha'_\bullet$ is a hypergroupoid action.

Part (I): For reference, diagram (17) shows the key portion of the lattice-diagram for $E'_{\bullet,1} = \mathrm{TORS}(K; \tilde{g}_\bullet)(E_{\bullet,1})$. Recall that $(y,x') \in E_{n-1} * \tilde{G}'_1 \hookrightarrow E_{n-1} \times G'_n$ if $g_{n-1}\alpha_{n-1}y = d_{n-1}x'$, and x' satisfies $(*n, 1)$. Similarly, an element of $E_{n-1} * \tilde{G}_2 * \tilde{G}'_1$ is $(y,x,x') \in E_{n-1} \times G_n \times G'_n$ if $g_{n-1}\alpha_{n-1}y = d_{n-1}x'$ and $\alpha_{n-1}y = d_{n-1}x$. (It then follows that $d_{n-1}g_nx = d_{n-1}x'$.)



(17)

Recall also that $D_0(y,x,x') = (y,x')$ and $D_1(y,x,x') = (yx, g_nx^{-1}x')$ and that $\tilde{\alpha}'_0 q_0(y,x') = d_nx'$. Now consider diagram (18).



(18)

In this diagram, $d_i(y,x') = d_iy = d_i(y,x,x')$, S_0 and S_1 are horizontal simplicial kernels, and the right and middle columns are exact. The left-most column is therefore also exact (a simple diagram chasing argument). We will define $\zeta: S_0 \to G'_n$ so that $\zeta D_0 = \zeta D_1$. This will determine α'_n in the quotient.

An element of S_0 is $((y_0, x'_0), \ldots, (y_n, x'_n))$ with $(y_i, x'_i) \in E_{n-1} * \tilde{G}'_1$ and

$(y_0, \ldots, y_n) \in E_n$. We will abbreviate this by $(\boldsymbol{y}, \boldsymbol{x}')$. Similarly, an element of S_1 is $(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{x}')$ where $(y_i, x_i, x_i') \in E_{n-1} * \tilde{G}_1 * \tilde{G}_1'$ for $i = 0, \ldots, n$ and $\boldsymbol{y} \in E_n$. Note that if $(\boldsymbol{y}, \boldsymbol{x}') \in S_0$ then for $0 \le i < j \le n-1$ one has

$$d_i x_j' = s_{n-2} d_i d_{n-1} x_j' = s_{n-2} d_i g_{n-1} \alpha_{n-1} y_j = s_{n-2} d_{j-1} g_{n-1} \alpha_{n-1} y_i = d_{j-1} x_i'.$$

Also, $d_i g_n \alpha_n \boldsymbol{y} = d_{n-1} x_i'$. Now let $(\boldsymbol{y}, \boldsymbol{x}') \in S_0$ and consider the matrix of G_{n+2}' shown in (19).

$$\begin{bmatrix} * & * & \cdots & x_0' & x_0' & * & * \\ * & * & \cdots & x_1' & x_1' & * & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & x_{n-2}' & x_{n-2}' & * & * \\ x_0' & x_1' & \cdots & x_{n-2}' & x_{n-1}' & g_n \alpha_n \boldsymbol{y} & t \\ x_0' & x_1' & \cdots & x_{n-2}' & x_{n-1}' & v & z \\ * & * & \cdots & * & g_n \alpha_n \boldsymbol{y} & v & x_n' \\ * & * & \cdots & * & t & z & x_n' \end{bmatrix} \begin{matrix} 0 \\ 1 \\ \\ n-2 \\ n-1 \\ n \\ n+1 \\ n+2 \end{matrix} \tag{19}$$

$R_i = i$-th row $= s_{n-2} x_i'$ for $0 \le i \le n-2$. Since $(x_0', \ldots, x_{n-2}', g_n \alpha_n \boldsymbol{y}, -)$ is in $\Lambda^{n+1}(n+1)(G_\bullet')$, a unique $t = [x_0', \ldots, x_{n-2}', g_n \alpha_n \boldsymbol{y}]$ exists, thus determining R_{n-1}. Similarly, v is determined in R_{n+1}. Finally, $z \in G_n'$ is determined as shown in R_n and R_{n+2}.

Now define $\zeta(\boldsymbol{y}, \boldsymbol{x}') = z$ by the equation:

$$x_n' = [s_{n-2} d_n x_0', \ldots, s_{n-2} d_n x_{n-2}', [x_0', x_1', \ldots, x_{n-1}', g_n \alpha_n \boldsymbol{y}], z].$$

Verification that $\zeta D_0 = \zeta D_1$. Recall that $D_0(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{x}') = (\boldsymbol{y}, \boldsymbol{x}')$ and $D_1(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{x}') = (\boldsymbol{y}\boldsymbol{x}, g(\boldsymbol{x})^{-1} \boldsymbol{x}')$ abbreviating

$$((y_0 x_0, g_n(x_0^{-1}) x_0'), \ldots, (y_n x_n, g_n(x_n^{-1}) x_n')).$$

Let

$$t = [x_0', \ldots, x_{n-1}', g_n \alpha_n \boldsymbol{y}],$$

$$x_n' = [s_{n-2} d_{n-1} x_0', \ldots, s_{n-2} d_{n-1} x_{n-2}' g_n \alpha_n \boldsymbol{y}, v]$$

$$= [s_{n-2} d_n x_0', \ldots, s_{n-2} d_n x_{n-2}', t, z],$$

and $z = [x_0', \ldots, x_{n-1}', v]$ as in the matrix above. The analogous matrix for $D_1(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{x}')$ will have

$$t' = [g_n(x_0^{-1}) x_0', \ldots, g_n(x_{n-1}^{-1}) x_{n-1}', g_n \alpha_n \boldsymbol{y}\boldsymbol{x}]$$

and will have

$$g_n(x_n^{-1}) x_n' = [s_{n-2} d_{n-1}(g_n(x_0^{-1}) x_0'), \ldots, s_{n-2} d_{n-1} g_n(x_{n-2}^{-1}) x_{n-2}', g_n \alpha_n \boldsymbol{y}\boldsymbol{x}, v']$$

$$= [s_{n-2} d_n g_n(x_0^{-1}) x_0', \ldots, s_{n-2} d_n g_n(x_{n-2}^{-1}) x_{n-2}', t', z']$$

$$= [s_{n-2} d_n x_0', \ldots, s_{n-2} d_n x_{n-2}', t', z'] \quad \text{(because } d_n g_n(x^{-1}) x' = d_n x').$$

Our goal then is to show $z=z'$.

Consider the matrix in G'_{n+2} given in (20).

$$\begin{bmatrix} * & \cdots & * & * & * & * \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & * & * \\ * & \cdots & * & t & t' & w' \\ * & \cdots & * & t & z & x'_n \\ * & \cdots & * & t' & z & u \\ * & \cdots & * & w' & x'_n & u \end{bmatrix} \begin{matrix} 0 \\ \\ n-2 \\ n-1 \\ n \\ n+1 \\ n+2 \end{matrix} \tag{20}$$

For $0 \le i \le n-2$, $R_i = s^2_{n-2} d_n x'_i$. R_n comes from the matrix defining $\zeta(\boldsymbol{y},\boldsymbol{x}')=z$. The hypergroupoid structure of $G'_{\bullet}$ then determines a unique w' in R_{n-1}, and a unique u in R_{n+1}. R_{n+2} is then also uniquely determined. It follows from a straightforward verification that $w', u \in \bar{G}'_1 \hookrightarrow G'_n$ and that $x'_n = w'u$ (reading from R_{n+2}).

Now compare the following elements from G'_{n+1}:

$$(s_{n-2} d_n x'_0, \dots, s_{n-2} d_n x'_{n-2}, t', z', g_n(x_n^{-1}) x'_n)$$

from R_{n+1} in the matrix above;

$$(s_{n-2} d_n x'_0, \dots, s_{n-2} d_n x'_{n-2}, t', z',, g_n(x_n^{-1}) x'_n)$$

from the matrix for $\zeta(\boldsymbol{yx}, g(\boldsymbol{x}^{-1})\boldsymbol{x}')$. This shows that $z=z'$ iff $u=g_n(x_n^{-1})x'_n$ iff $w'=g_n x_n$. So we will now verify $w'=g_n x_n$.

Now $t'=[g_n(x_0^{-1})x'_0, \dots, g_n(x_{n-1}^{-1})x'_{n-1}, g_n \alpha_n \boldsymbol{yx}]$. By Corollary 3.4.2,

$$t'=[x'_0, \,.\,, x'_{n-1}, [g_n(x_0^{-1}), \dots, g_n(x_{n-1}^{-1}), g_n \alpha_n \boldsymbol{yx}]].$$

Using Proposition 3.8.4 we get

$$g_n \alpha_n \boldsymbol{yx} = [g_0 x_0, \dots, g_n x_{n-1}, g_n \alpha_n (y_0, \dots, y_{n-1}, y_n x_n)].$$

So again by Corollary 3.4.2,

$$\begin{aligned} [g_n x_0^{-1}, \dots, g_n x_{n-1}^{-1}, g_n \alpha_n \boldsymbol{yx}] &= [g_0 x_0^{-1}, \dots, g_n x_{n-1}^{-1}, [g_0 x_0, \dots, g_n x_{n-1}, g_n P]] \\ &= [1, \dots, 1, P] \end{aligned}$$

where 'P' stands for $\alpha_n(y_0, \dots, y_{n-1}, y_n x_n)$. Once again applying Corollary 3.4.2 we obtain:

$$t'=[x'_0, \dots, x'_{n-1}, g_n P].$$

Consider the matrix in G'_{n+2} given in (21).

$$\begin{bmatrix} * & * & \cdots & x'_0 & x'_0 & * & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & x'_{n-2} & x'_{n-2} & * & * \\ x'_0 & x'_1 & \cdots & x'_{n-2} & x'_{n-1} & g_n \alpha_n \boldsymbol{y} & t \\ x'_0 & x'_1 & \cdots & x'_{n-2} & x'_{n-1} & g_n P & t' \\ * & * & \cdots & * & g_n \alpha_n \boldsymbol{y} & g_n P & w' \\ * & * & \cdots & * & t & t' & w' \end{bmatrix} \begin{matrix} 0 \\ \vdots \\ n-2 \\ n-1 \\ n \\ n+1 \\ n+2 \end{matrix} \tag{21}$$

For $0 \leq i \leq n-2$, $R_i = s_{n-2}x'_i$. R_{n-1} is from $\zeta(\boldsymbol{y}, \boldsymbol{x}')$. R_n is from the expression for t' we just derived and R_{n+2} is from the previous matrix in which w' was defined. R_{n+1}, which is then uniquely determined, shows:

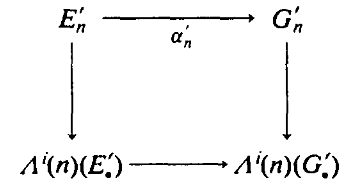
$$w' = [s_{n-2}d_{n-1}x'_0, \ldots, s_{n-2}d_{n-1}x'_{n-2}, g_n\alpha_n\boldsymbol{y}, g_nP].$$

By Proposition 3.8.4 (part (b))

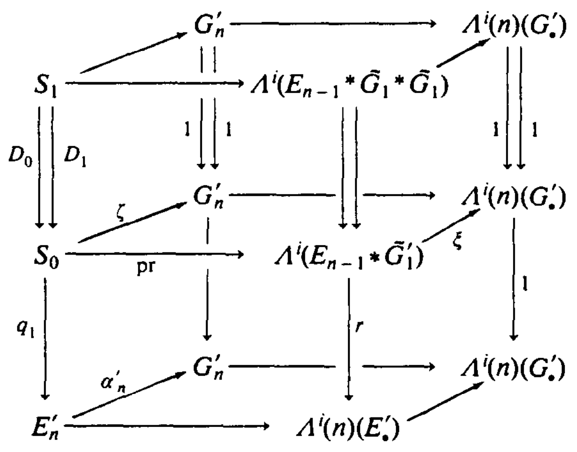
$$x_n = [s_{n-2}\alpha_{n-1}y_0, \ldots, s_{n-2}\alpha_{n-1}y_{n-2}, \alpha_n\boldsymbol{y}, P].$$

Then since $g_ns_{n-2}\alpha_{n-1}y_i = s_{n-2}d_{n-1}x_i$, we have $g_nx_n = w'$. This completes the verification of $\zeta D_0 = \zeta D_1$.

Part (II): In order to show that $\alpha'_\bullet$ is a hypergroupoid action we will show that



is a pullback for each i and then apply Lemma 3.6.1. We will work with diagram (22).



(22)

In diagram (22), $\Lambda^i(E_{n-1} * \tilde{G}'_1)$ denotes the open i-boxes for

$$E_{n-1} * \tilde{G}'_1 \underset{d_{n-1}}{\overset{d_0}{\underset{\cdots}{\rightrightarrows}}} E_{n-2}.$$

Recall that $d_j(y, x') = d_jy$. Similarly, $\Lambda^i(E_{n-1} * \tilde{G}_1 * \tilde{G}'_1)$ denotes the open i-boxes for $E_{n-1} * \tilde{G}_1 * \tilde{G}'_1 \rightrightarrows E_{n-2}$. The maps ξ and q_1 are defined by:

$$\xi((y_0, x'_0), \ldots, -, \ldots, (y_n, x'_n)) = (d_nx'_0, \ldots, -, \ldots, d_nx'_n),$$

$$q_1((y_0, x'_0), \ldots, (y_n, x'_n)) = (q_0(y_0, x'_0), \ldots, q_0(y_n, x'_n)).$$

A straightforward diagram chase shows that

$$\Lambda^i(E_{n-1} * \tilde{G}_1 * \tilde{G}'_1) \rightrightarrows \Lambda^i(E_{n-1} * \tilde{G}'_1) \to \Lambda^i(n)(E'_\bullet)$$

is exact. All the other columns of (22) are exact. The rear plane of (22) is (trivially) a pullback of exact sequences. After we show that the horizontal squares involving S_0 and S_1 are pullbacks, it will follow that the front plane of (22) is also a pullback of exact sequences. The hypotheses of Lemma 2.4.5 then hold for (22) and we can conclude that the bottom of (22) is a pullback. That will complete part (II) of the proof.

Suppose

$$\begin{array}{ccc} W_0 & \longrightarrow & G'_n \\ \downarrow & & \downarrow \\ \Lambda^i(E_n * \tilde{G}'_1) & \longrightarrow & \Lambda^i(n)(G'_\bullet) \end{array}$$

is a pullback. We may apply the Barr Embedding Theorem (Section 1.3) and assume this square is in $\mathscr{S}$. We will show $W_0 \cong S_0$. (A similar argument works for S_1.) An element of W_0 is

$$((y_0, x'_0), \ldots, -, \ldots, (y_n, x'_n), z) \in \Lambda^i(E_{n-1} * \tilde{G}'_1) \times G'_n$$

where $d_j z = d_n x'_j$. Since $E_\bullet$ is aspherical, there is a $\bar{y}_i \in E_{n-1}$ such that $(y_0, \ldots, \bar{y}_i, \ldots, y_n) \in E_n$.

Case $i \neq n$: A unique $\upsilon \in G'_n$ is defined by the hypergroupoid structure from

$$(s_{n-2}d_{n-1}x'_0, \ldots, s_{n-2}g_{n-1}\alpha_{n-1}\bar{y}_i, \ldots, s_{n-2}d_{n-1}x'_{n-2}, g_n\alpha_n y, \upsilon, x') \in G'_{n+1}.$$

A unique $\bar{x}'_i \in \tilde{G}'_1 \hookrightarrow G'_n$ is then defined by the hypergroupoid structure from

$$(x'_0, \ldots, \bar{x}'_i, \ldots, x'_{n-1}, \upsilon, z) \in G'_{n+1}.$$

Then $((y_0, x'_0), \ldots, (\bar{y}_i, \bar{x}'_i), \ldots, (y_n, x'_n)) \in S_0$. This provides a map $W_0 \to S_0$ which is an inverse to the canonical $S_0 \to W_0$.

Case $i = n$: Define υ immediately by $(x'_0, \ldots, x'_{n-1}, \upsilon, z) \in G'_{n+1}$. Then define x'_n by

$$(s_{n-2}d_{n-1}x'_0, \ldots, s_{n-2}d_{n-1}x'_{n-2}, g_n\alpha_n y, \upsilon, x'_n) \in G'_{n+1}.$$

Again $S_0 \cong W_0$. This completes part (II).

We now complete the proof of the theorem by observing that the functoriality of $\mathrm{TORS}(X;\ g_\bullet)$ follows from that of the extension-of-the-structural-groupoid construction (Theorem 2.4.1) since $\mathrm{TORS}(X;\ g_\bullet)$ was defined using that construction on the attached 1-torsor of $E_\bullet$. Similarly, the isomorphism

$$\mathrm{TORS}(X;\ g'_\bullet g_\bullet) \cong \mathrm{TORS}(X;\ g'_\bullet)\mathrm{TORS}(X;\ g_\bullet)$$

follows from the analogous one in Theorem 2.4.1. □

Theorem 4.2. *Let $g_\bullet : G_\bullet \to G'_\bullet$ be a map of n-dimensional hypergroupoids and $X' \to X$ an arbitrary map. Then*

$$\begin{array}{ccc} \mathrm{TORS}(X;\, G_\bullet) & \longrightarrow & \mathrm{TORS}(X;\, G'_\bullet) \\ \downarrow & & \downarrow \\ \mathrm{TORS}(X';\, G_\bullet) & \longrightarrow & \mathrm{TORS}(X';\, G'_\bullet) \end{array}$$

commutes up to isomorphism.

Proof. Both composites are equal on the $(n-2)$-truncation of a torsor $E_\bullet \in \mathrm{TORS}(X;\, G_\bullet)$. They are isomorphic on the attached 1-torsor level by Theorem 2.4.2. □

We will conclude this chapter with two additional facts about $\mathrm{TORS}(X;\, g_\bullet)$.

Proposition 4.3. *Let $g_\bullet : G_\bullet \to G'_\bullet$ be a map of n-dimensional hypergroupoids, let $\alpha_\bullet : E_\bullet \to G_\bullet$ be a torsor over X and let $E'_\bullet = \mathrm{TORS}(X;\, g_\bullet)$. Then there is a $G_\bullet$-equivariant map $h_\bullet : E_\bullet \to E'_\bullet$ such that*

$$\begin{array}{ccc} G_\bullet & \xrightarrow{\;g_\bullet\;} & G'_\bullet \\ {\scriptstyle \alpha_\bullet}\uparrow & & \uparrow{\scriptstyle \alpha'_\bullet} \\ E_\bullet & \xrightarrow{\;h_\bullet\;} & E'_\bullet \end{array}$$

commutes.

(Compare with Lemma 2.4.3).

Proof. For $0 \leq m \leq n-2$, set $h_m = 1_{E_m}$. At dimension $n-1$ define h_{n-1} by $h_{n-1}y = q_0(y, s_{n-1}g_{n-1}\alpha_{n-1}y)$. See diagram (18) and recall that $(y, s_{n-1}g_{n-1}\alpha_{n-1}y) \in E_{n-1} * \tilde{G}'_1$. This definition applies Lemma 2.4.3 at the attached 1-torsor level. The simplicial map $h_\bullet$ is then determined for all $m \geq n$ since E_m and E'_m are simplicial kernels for $m \geq n$. We must show $g_\bullet\alpha_\bullet = \alpha'_\bullet h_\bullet$. Now $g_m\alpha_m = \alpha'_m h_m$ for $m \leq n-2$ since, by definition, $\alpha'_m = g_m\alpha_m$ and $h_m = 1$. Now let $y = (y_0, \ldots, y_n) \in E_n$. Abbreviate $s_{n-1}\alpha_{n-1}y_i$ by 1_i and $g_n 1_i = s_{n-1}g_{n-1}\alpha_{n-1}y_i$ by $1'_i$. Then

$$\begin{aligned} \alpha'_n h_n y &= \alpha'_n(q_0(y_0, 1'_0), \ldots, q_0(y_n, 1'_n)) \\ &= \alpha'_n q_1((y_0, 1'_0), \ldots, (y_n, 1'_n)) \\ &= \zeta((y_0, 1'_0), \ldots, (y_n, 1'_n)) = z \end{aligned}$$

as defined in the proof of Theorem 4.1. So we need to show that $g_n\alpha_n y = z = \zeta((y_0, 1'_0), \ldots, (y_n, 1'_n))$. Now z satisfies the hypergroupoid equation

$$1'_n = [s_{n-2}d_n 1'_0, \ldots, s_{n-2}d_n 1'_{n-2}, [1'_0, \ldots, 1'_{n-1}, g_n\alpha_n y], z].$$

By Proposition 3.8.4,

$$\alpha_n y = \alpha_n(y_0 1_0, \ldots, y_n 1_n) = [1_0, \ldots, 1_{n-1}, \alpha_n(y_0, \ldots, y_{n-1}, y_n)].$$

Hence $g_n \alpha_n y = [1'_0, \ldots, 1'_{n-1}, g_n \alpha_n y]$. We thus have

$$(s_{n-2} d_n 1'_0, \ldots, s_{n-2} d_n 1'_{n-2}, g_n \alpha_n y, z, 1'_n) \in G'_{n+1}.$$

But the hyper-unit identity $s_{n-1} g_n \alpha_n y \in G'_{n+1}$ implies that $z = d_n s_{n-1} g_n \alpha_n(y) = g_n \alpha_n y$. □

Corollary 4.4. *Let $g_\bullet : G_\bullet \to G'_\bullet$ be a map of n-dimensional hypergroupoids and suppose also that $g_\bullet$ is an exact fibration in dimensions $\geq n-1$. If $\alpha_\bullet : E_\bullet \to G_\bullet$ is a torsor over X, then the composite $g_\bullet \alpha_\bullet : E_\bullet \to G'_\bullet$ is a torsor under $G'_\bullet$ over X. Furthermore,* TORS$(X;\ g_\bullet)(E_\bullet)$ *is $g_\bullet \alpha_\bullet : E_\bullet \to G'_\bullet$.*

Proof. It is easy to see that $g_\bullet \alpha_\bullet$ is an exact fibration in dimensions $\geq n-1$ and hence is a hypergroupoid action. The condition COSK$^{n-1}(E_\bullet)$ still holds, obviously, as does asphericity, and so $g_\bullet \alpha_\bullet$ defines a torsor in TORS$(X;\ G'_\bullet)$. Let $E'_\bullet =$ TORS$(X;\ g_\bullet)(E_\bullet)$. By Proposition 4.3 we have a diagram

$$\begin{array}{ccc} G_\bullet & \xrightarrow{\ g_\bullet\ } & G'_\bullet \\ {\scriptstyle \alpha_\bullet}\uparrow & & \uparrow{\scriptstyle \alpha'_\bullet} \\ E_\bullet & \xrightarrow[\ h_\bullet\ ]{} & E'_\bullet \end{array}$$

$h_\bullet$ is a torsor map in TORS$(X;\ G'_\bullet)$ since $g_\bullet \alpha_\bullet$ defines a torsor. On the attached 1-torsor level, the $G'_\bullet$-equivariant map $h_\bullet$ is an isomorphism in TORS$(\Delta^\bullet(n-1)(E_\bullet);\ \tilde{G}'_1)$ since $E_{\bullet,1} \to E'_{\bullet,1}$ is the pullback of an identity map. Hence $\alpha'_\bullet = g_\bullet \alpha_\bullet$. □

5. Torsors and cohomology groups

In this chapter we will consider torsors under the n-dimensional hypergroupoid $K(A,n)$ where A is an abelian group object in the exact category $\mathscr{C}$. The equations derived in Chapter 3 which characterized the interplay between n-dimensional torsors and their attached 1-torsors simplify in this case because $K(A,n)$ has only one degenerate n-simplex, namely $0 \in A$.

A torsor under $K(A,n)$ has a substructure, called its *fiber*, for which there is no clear analog in the general hypergroupoid case. The fiber is an $(n-1)$-dimensional hypergroupoid. It plays a key part in establishing the long exact sequence of cohomology and also shows how an n-torsor under $K(A,n)$ can be regarded as a 1-torsor.

The category TORS$(X;\ K(A,n))$ has special properties not possessed by TORS$(X;\ G_\bullet)$ for general $G_\bullet$. First, it contains a distinguished torsor and thus is non-empty. Second, it has a rather simple connected components structure. Third, the addition map for A yields a functorial associative and commutative binary product defined on torsors which determines a way of adding connected components. The result is an abelian group of connected components which is by definition the n-th cohomology group of X with coefficients in A.

5.1. n-torsors under A

Let A be an abelian group object. Denote TORS(X; $K(A,n)$) by TORS^n(X; A) for short. A torsor in this category is an 'n-torsor under A (over X)'. The results of Chapter 3, as they apply to $K(A,n)$ and an n-torsor $\alpha_\bullet: E_\bullet \to K(A,n)$ include:

1. The associated groupoid of $K(A,n)$ is $K(A,1)$, i.e. the group A itself.
2. $E_\bullet$ is a 1-torsor iff A acts principally and effectively on E_0 with quotient $p: E_0 \to X$, the map onto the 'orbits'.
3. The attached 1-torsor of $E_\bullet$ (if $n>1$) is a 1-torsor under A over $\Delta^\bullet(n-1)(E_\bullet)$.
4. (Notation). Given $a_0, \ldots, a_n \in A$, then A.S.$(a_0, \ldots, a_n)$ abbreviates the alternating sum $a_n - a_{n-1} + \cdots + (-1)^n a_0$. In the hypergroupoid structure of $K(A,n)$, $[a_0, \ldots, a_n] = \text{A.S.}(a_0, \ldots, a_n)$. Since all degenerate simplices of $K(A,n)$ are '0', the equations in Proposition 3.8.4 take the form

$$\alpha_n(y_0 a_0, \ldots, y_n a_n) = \alpha_n(y_0, \ldots, y_n) + \text{A.S.}(a_0, \ldots, a_n).$$

5.2. The fiber of an n-torsor

Let $1_\bullet$ denote the simplicial object consisting of 1 at every dimension and let $e_\bullet: 1_\bullet \to K(A,n)$ be the simplicial map defined by setting e_n to be $0: 1 \to A$ in dimension n.

Definition. Let $\alpha_\bullet: E_\bullet \to K(A,n)$ be an n-torsor. The pullback simplicial object

$$\begin{array}{ccc} G_\bullet(E_\bullet) & \longrightarrow & E_\bullet \\ \big\downarrow & & \big\downarrow{\scriptstyle \alpha_\bullet} \\ 1_\bullet & \longrightarrow & K(A,n) \end{array}$$

is called the *fiber* of $E_\bullet$.

This concept defines a functor on TORS^n(X; A). For $0 \le m \le n-1$, $G_m(E_\bullet) = E_m$ and $G_n(E_\bullet)$ consists of all $y \in E_n$ such that $\alpha_n y = 0$. Since $E_n \cong \Lambda^i(n)(E_\bullet) \times A$ with α_n the projection on A, it follows that $G_n(E_\bullet) \cong \Lambda^i(n)(E_\bullet)$. Similarly, $G_m(E_\bullet) \cong \Lambda^i(m)(E_\bullet)$ for every $m > n$. This proves:

Proposition 5.2.1. *$G_\bullet(E_\bullet)$ is an $(n-1)$-dimensional hypergroupoid.* □

5.3. *Every n-torsor under A is a* 1*-torsor*

Let $\alpha_\bullet : E_\bullet \to K(A,n)$ be an n-torsor with attached 1-torsor

$$E_{n-1} \times A \rightrightarrows E_{n-1} \to \Delta^\bullet(n-1)(E_\bullet).$$

The object E_{n-1} has an $(n-1)$-dimensional hypergroupoid structure (as part of $G_\bullet(E_\bullet)$). The group A is an $(n-1)$-dimensional hypergroupoid also (from $K(A,n-1)$) as is $\Delta^\bullet(n-1)(E_\bullet)$ from being part of $\mathrm{COSK}^{n-2}(E_\bullet)$. The action of A on E_{n-1} respects these hypergroupoid structures. Also, one can reconstruct from such a 1-torsor the n-torsor whose attached 1-torsor it is.

Theorem 5.3.1. (i) *If $\alpha_\bullet : E_\bullet \to K(A,n)$ is an n-torsor, then its attached* 1*-torsor is a* 1*-torsor in the category of $(n-1)$-dimensional hypergroupoids.*

(ii) *Let $n \geq 1$ and let $G_\bullet$ be an $(n-1)$-dimensional hypergroupoid which is augmented over X and aspherical. If*

$$G_{n-1} \times A \rightrightarrows G_{n-1} \to \Delta^\bullet(n-1)(G_\bullet)$$

is a 1*-torsor in* $\mathrm{Hypgpd}_{n-1}(\mathscr{C})$ *then $E_\bullet = \mathrm{COSK}^{n-1}(G_\bullet)$ is an n-torsor under A over X whose fiber is $G_\bullet$ and whose attached* 1*-torsor is the given one.*

Proof. (i) The map $E_{n-1} \dot{\to} \Delta^\bullet(n-1)(E_\bullet)$ is obviously an $(n-1)$-dimensional hypergroupoid map as is the projection

$$E_{n-1} \times A \xrightarrow{p_0} E_{n-1}.$$

As for the other map

$$E_{n-1} \times A \xrightarrow[p_1]{} E_{n-1},$$

we must show that the following diagram commutes for all i and j:

$$\begin{array}{ccc}
G_n(E_\bullet) \times K(A,n-1)_n & \xrightarrow{\ p\ } & G_n(E_\bullet) \\
\downarrow d_i & & \downarrow d_i \\
E_{n-1} \times A & \xrightarrow[p_1]{} & E_{n-1} \\
\downarrow d_j & & \downarrow d_j \\
E_{n-2} & \xrightarrow[1]{} & E_{n-2}
\end{array}$$

An element of $G_n(E_\bullet) \times K(A,n-1)_n$ is $(y_0, \ldots, y_n, a_0, \ldots, a_0)$ where $\alpha_n \boldsymbol{y} = 0$ and $\mathrm{A.S.}(a_0, \ldots, a_n) = 0$. The map $\boldsymbol{p}$ sends $(\boldsymbol{y}, \boldsymbol{a})$ to $(y_0 a_0, \ldots, y_n a_n)$. The top square then obviously commutes. The bottom square commutes because $d_j y = d_j(ya)$ for all j.

(ii) $E_\bullet$ is isomorphic to $\mathrm{COSK}^{n-1}(E_\bullet)$ by definition and is aspherical over X. We need to define $\alpha_\bullet : E_\bullet \to K(A,n)$ (and it suffices to do so at dimension n) and show that it is a hypergroupoid action. First observe that at dimension n the action of $K(A,n)$ on $G_\bullet$ sends $(y_0,\dots,y_n,a_0,\dots,a_n)$ to $(y_0a_0,\dots,y_na_n)$. Since $y_n=[y_0,\dots,y_{n-1}]$ and

$$a_n=[a_0,\dots,a_{n-1}]=\mathrm{A.S.}(a_0,\dots,a_{n-1})$$

we have

$$[y_0a_0,\dots,y_{n-1}a_{n-1}]=y_na_n=[y_0,\dots,y_{n-1}]\mathrm{A.S.}(a_0,\dots,a_{n-1}).$$

Thus: $(y_0,\dots,y_{n-1},y_na)\in G_n$ iff, for each i, $(y_0,\dots,y_i(-1)^{n-i}a,\dots,y_n)\in G_n$. Now suppose $(y_0,\dots,y_n)\in E_n=\Delta^\bullet(n)(G_\bullet)$. Then the exactness of

$$G_{n-1}\times A \rightrightarrows G_{n-1}\to \Delta^\bullet(n-1)(G_\bullet)$$

implies $d_i([y_0,\dots,y_{n-1}])=d_iy_n$ for all i and therefore that $y_n=[y_0,\dots,y_{n-1}]a$ for some unique $a\in A$. Now define $\alpha_n(y_0,\dots,y_n)=a$ iff $y_n=[y_0,\dots,y_{n-1}]a$. This definition of α_n is forced by Proposition 3.8.4 together with the first observation above. To show that $\alpha_\bullet$ is a hypergroupoid action we must show $E_n\cong\Lambda^i(n)(E_\bullet)\times A$. The map $E_n\to\Lambda^i(n)(E_\bullet)\times A$ defined by

$$(y_0,\dots,y_n)\mapsto(y_0,\dots,-,\dots,y_n,\alpha_ny)$$

has as its inverse the map

$$(y_0,\dots,-,\dots,y_n,a)\to(y_0,\dots,y_i(-1)^{n-i}a,\dots,y_n)$$

where y_i is uniquely determined by the hypergroupoid structure of $G_\bullet$. It is immediate from this construction that $G_\bullet$ is the fiber of $E_\bullet$ and that $E_{\bullet,1}$ is the originally given 1-torsor. □

Corollary 5.3.2. *Suppose $\varphi_\bullet : E_\bullet \to E'_\bullet$ is a simplicial map between n-torsors under A. Then $\varphi_\bullet$ is an n-torsor map iff $\varphi_\bullet$ restricts to a map between the fibers.*

Proof. If $\varphi_\bullet$ restricts to a map between the fibers, then

$$\varphi_n(y_0,\dots,-,\dots,y_n,a)=(\varphi_{n-1}y_0,\dots,-,\dots,\varphi_{n-1}y_n,a)$$

and thus restricts to a 1-torsor map in $\mathrm{Hypgpd}_{n-1}(\mathscr{C})$. It is then clear from Theorem 5.3.1 that $\varphi_\bullet$ is an n-torsor map. □

5.4. Quasi-split torsors

The canonical map $d_\bullet : \mathrm{DEC}(K(A,n))\to K(A,n)$ is a hypergroupoid action (see Section 3.6); in fact, it is an n-torsor over 1 under A. The attached 1-torsor is

$$1 \longleftarrow A \underset{+}{\leftleftarrows} A\times A,$$

the group A acting on itself by right translation.

Definition. An n-torsor is *quasi-split* if its attached 1-torsor is split.

Denote $\mathrm{DEC}(K(A,n))$ by $K_{\#}(A,n)$ for short. It is the 'canonical' quasi-split torsor. It is also split as a simplicial object. Generally, a quasi-split torsor is not split however.

Recall (from Proposition 3.8.1) that any map $E_{\bullet,\mathrm{tr}} \to \mathrm{Tr}^{n-2}K_{\#}(A,n)$ can be extended to an n-torsor map $E_{\bullet} \to K_{\#}(A,n)$ of torsors under A. The attached 1-torsor of $E_{\bullet}$ is the pullback torsor of that of $K_{\#}(A,n)$ and is thus split. That is, $E_{\bullet}$ is quasi-split. This characterizes being quasi-split.

Proposition 5.4.1. *$\alpha_{\bullet}: E_{\bullet} \to K(A,n)$ is quasi-split iff $\alpha_{\bullet}$ factors through $K_{\#}(A,n) \to K(A,n)$.*

Proof. If $\alpha_{\bullet}$ factors through $K_{\#}(A,n)$, then $E_{\bullet}$ is quasi-split (Proposition 3.8.1). Conversely, if $E_{\bullet}$ is quasi-split then $E_{n-1} \cong \Delta^{\bullet}(n-1)(E_{\bullet}) \times A$ and E_n, as a simplicial kernel, has elements of the form: $((y_0, a_0), \ldots, (y_n, a_n))$ where $y_i = (y_{i0}, \ldots, y_{in-1}) \in \Delta^{\bullet}(n-1)(E_{\bullet})$ and $d_i y_j = y_{ji} = y_{i,j-1} = d_{j-1} y_i$ for $i<j$. Define $E_{\bullet} \to K_{\#}(A,n)$ at dimension n by sending $((y_0, a_0), \ldots, (y_n, a_n))$ to $(a_0, \ldots, a_n)$. Since $K_{\#}(A,n)_n \to A$ sends $(a_0, \ldots, a_n)$ to $\mathrm{A.S.}(a_0, \ldots, a_n)$ we must show that

$$\alpha_n((y_0, a_0), \ldots, (y_n, a_n)) = \mathrm{A.S.}(a_0, \ldots, a_n).$$

Now the A-action on the attached 1-torsor of E is $(y,a)a' = (y, a+a')$. Then

$$\begin{aligned}\alpha_n((y_0, a_0), \ldots, (y_n, a_n)) &= \alpha_n((y_0, 0)a_0, \ldots, (y_n, 0)a_n) \\ &= \alpha_n((y_0, 0), \ldots, (y_n, 0)) + \mathrm{A.S.}(a_0, \ldots, a_n).\end{aligned}$$

To see that $\alpha_n(\ldots, (y_i, 0), \ldots) = 0$ consider the matrix in E_{n+1} whose bottom ($(n+1)$-st) row is $(\ldots, (y_i, 0), \ldots)$ and whose i-th row for $0 \le i \le n$ is $s_{n-1}(y_i, 0)$. Then

$$\begin{aligned}\alpha_n(\ldots, (y_i, 0), \ldots) &= d_{n+1}\alpha_{n+1}(\text{matrix}) \\ &= \mathrm{A.S.}(\alpha_n s_{n-1}(y_0, 0), \ldots, \alpha_n s_{n-1}(y_n, 0)) = 0. \qquad \square\end{aligned}$$

Corollary 5.4.2. *$\mathrm{TORS}^n(X; A)$ is non-empty.*

Proof. For any X there is the constant truncated complex consisting of X at every dimension and with all face and degeneracy maps 1_X. There is a unique truncated map from this complex to $\mathrm{Tr}^{n-2}K_{\#}(A,n)$ which extends to an n-torsor map $E_{\bullet}^{\#} \to K_{\#}(A,n)$. This is a quasi-split torsor over X and $E_{\bullet}^{\#} \to K(A,n)$ is unique by Proposition 5.4.1. $\square$

Remark. If $E_{\bullet} \to E'_{\bullet}$ is a torsor map and $E'_{\bullet}$ is quasi-split, then so is $E_{\bullet}$. But one cannot conclude that $E'_{\bullet}$ is quasi-split from $E_{\bullet}$ being quasi-split.

5.5. *Connected components of* TORS$^n(X;\, A)$*: preliminary facts*

Definition. A *connected component* of $\mathscr{C}$ is an equivalence class of the equivalence relation generated by the following relation: $X \sim Y$ iff $\mathscr{C}(X, Y) \neq \emptyset$.

Let $[X]$ denote the equivalence class represented by X and TORS$^n[X;\, A]$ the class of connected components of TORS$^n(X;\, A)$. Note that $[X] = [Y]$ iff one has a series of maps

$$X \to A_0 \leftarrow A_1 \to \cdots \to A_n \leftarrow Y.$$

We will see later that two torsors in the same component can always be linked by one intervening pair of maps.

The following two facts will suffice to prove that TORS$^n[X;\, A]$ is an abelian group.

Lemma 5.5.1. *Given any* $[E_\bullet]$ *and* $[E'_\bullet]$ *in* TORS$^n[X;\, A]$*, one can find representatives from each component having equal* $(n-2)$*-truncations.*

Proof. Choose any representatives $E_\bullet$ and $E'_\bullet$ and form the pullback of $(n-2)$-truncated complexes

$$\begin{array}{ccc} \bar{E}_{\bullet,\mathrm{tr}} & \longrightarrow & \mathrm{Tr}^{n-2}(E'_\bullet) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Tr}^{n-2}(E_\bullet) & \longrightarrow & \mathrm{Con}(X) \end{array}$$

where $\mathrm{Con}(X)$ is the simplicial object consisting of X at every dimension with all the face and degeneracy maps equal to 1_X. $E_m \to X$ is pd_0^m.

$\bar{E}_{\bullet,\mathrm{tr}}$ is aspherical. (This is easily verified in sets. Apply the Embedding Theorem.)

Then $\bar{E}_{\bullet,\mathrm{tr}} \to \mathrm{Tr}^{n-2}(E_\bullet)$ and $\bar{E}_{\bullet,\mathrm{tr}} \to \mathrm{Tr}^{n-2}(E'_\bullet)$ extend to torsors maps $\bar{E}_\bullet \to E_\bullet$ and $\bar{E}'_\bullet \to E'_\bullet$ (by Proposition 3.8.1) where $\bar{E}_\bullet$ and $\bar{E}'_\bullet$ have equal $(n-2)$-truncations by construction and $[\bar{E}_\bullet] = [E]$ and $[\bar{E}'_\bullet] = [E'_\bullet]$. $\square$

Lemma 5.5.2. *If* $E_\bullet$ *is quasi-split then* $[E_\bullet] = [E^\#_\bullet]$ *where* $E^\#_\bullet$ *is the canonical quasi-split torsor over X defined in Corollary* 5.4.2.

Proof. For $m < n-1$, we have maps $pd_0^m : E_m \to E_m^\# = X$. Since $E_\bullet$ is quasi-split, $E_{n-1} \cong \Delta^\bullet(n-1)(E_\bullet) \times A$. Define $E_{n-1} \to X \times A$ by $(y, a) \mapsto (pd_0^{n-2} y_0, a)$. We thus have $E_\bullet \to E^\#_\bullet$ which, since it restricts to a map of the fibers, is by Corollary 5.3.2 a torsor map. Therefore $[E_\bullet] = [E^\#_\bullet]$. $\square$

5.6. Abelian group structure on $\mathrm{TORS}^n[X;\ A]$. *Functoriality*

We will now define a binary operation on torsors which will determine an abelian group addition of connected components.

Definition. Let $E_\bullet, E'_\bullet \in \mathrm{TORS}^n(X;\ A)$. Let $\Delta_X : X \to X \times X$ denote the diagonal map and let $+ : A \times A \to A$ be the addition map for A.

Set $E_\bullet \otimes E'_\bullet = \mathrm{TORS}^n(X;\ +)\mathrm{TORS}^n(\Delta_X;\ A^2)(E_\bullet \times E'_\bullet)$.

Remarks. $E_\bullet \times E'_\bullet$ is the product simplicial object formed of products dimension by dimension. $E_\bullet \times E'_\bullet \in \mathrm{TORS}^n(X^2;\ A^2)$. The addition map $+$ is a homomorphism since A is abelian. Since '$\otimes$' is defined by a composite of functors, it is functorial in each variable and, in particular, respects connected components.

Lemma 5.6.1. *If* $[E_\bullet \otimes E'_\bullet] = [E''_\bullet]$ *in* $\mathrm{TORS}^n[X;\ A]$, *one may choose representatives* $E_\bullet, E'_\bullet, E''_\bullet$ *with equal* $(n-2)$*-truncations so that* $E''_{\bullet,1} \cong E_{\bullet,1} \otimes E'_{\bullet,1}$ *on the attached* 1*-torsor level in* $\mathrm{TORS}^1(K; A)$ *where* $K = \Delta^\bullet(n-2)(E_\bullet) = \Delta^\bullet(n-2)(E'_\bullet) = \Delta^\bullet(n-2)(E''_\bullet)$.

Proof. Choose, according to Lemma 5.5.1, representatives $E_\bullet$ and $E'_\bullet$ with equal $(n-2)$-truncations, say $\bar{E}_{\bullet,\mathrm{tr}}$. Let $E_\bullet * E'_\bullet$ denote $\mathrm{TORS}^n(\Delta_X;\ A^2)(E_\bullet \times E'_\bullet)$.

$$\begin{array}{ccc} E_\bullet * E'_\bullet & \longrightarrow & E'_\bullet \\ \downarrow & & \downarrow \\ E_\bullet & \longrightarrow & X \end{array}$$

Then $\bar{E}_{\bullet,\mathrm{tr}} \to \mathrm{Tr}^{n-2}(E_\bullet * E'_\bullet)$ (the diagonal map over X), extends to a torsor map $E^*_\bullet \to E_\bullet * E'_\bullet$. We thus have

$$E''_\bullet = \mathrm{TORS}^n(X;\ +)(E^*_\bullet) \to E_\bullet \otimes E'_\bullet,$$

showing that $[E''_\bullet] = [E_\bullet \otimes E'_\bullet]$. $E''_\bullet$ clearly has the same $(n-2)$-truncation as $E_\bullet$ and $E'_\bullet$ and it is easily seen that $E''_{\bullet,1} = E_{\bullet,1} \otimes E'_{\bullet,1}$ in $\mathrm{TORS}^1(\Delta^\bullet(n-1)(E_\bullet);\ A)$. □

Proposition 5.6.2. *For torsors in* $\mathrm{TORS}^n(X;\ A)$ *the following statements are true.*

(i) $E_\bullet \otimes E'_\bullet = E'_\bullet \otimes E_\bullet$.

(ii) $[E_\bullet \otimes E^s_\bullet] = [E_\bullet]$ *where* $E^s_\bullet$ *is any quasi-split torsor.*

(iii) $[E_\bullet \otimes (E'_\bullet \otimes E''_\bullet)] = [(E_\bullet \otimes E'_\bullet) \otimes E''_\bullet]$.

Proof. (i) Obvious from the definition.

(ii) By Lemma 5.6.1 we can assume that $E_\bullet$, $E^s_\bullet$ and $E'_\bullet = E_\bullet \otimes E^s_\bullet$ have equal $(n-2)$-truncations and that $E'_{\bullet,1} = E_{\bullet,1} \otimes E^s_{\bullet,1}$. The assertion is thus reduced to dimension 1; its straight-forward verification is left to the reader.

(iii) Form the torsor $E_\bullet \times E'_\bullet \times E''_\bullet \in \mathrm{TORS}^n(X^3;\ A^3)$ and pull back along

$\Delta_{X_3}: X \to X^3$ to obtain $E_\bullet * E'_\bullet * E''_\bullet \in \mathrm{TORS}^n(X; A^3)$. The addition homorphism $\sigma: A^3 \to A$ yields

$$E_\bullet \otimes E'_\bullet \otimes E''_\bullet = \mathrm{TORS}^n(X; \sigma)(E_\bullet * E'_\bullet * E''_\bullet).$$

It suffices to compare this to $E_\bullet \otimes (E'_\bullet \otimes E''_\bullet)$. By Lemma 5.6.1 one can arrange for both torsors to have equal $(n-2)$-truncations. This reduces the comparison to the attached 1-torsors level. Again the details are left to the reader. □

Remark. In fact, it can be shown that

$$E_\bullet \otimes (E'_\bullet \otimes E''_\bullet) \cong (E_\bullet \otimes E'_\bullet) \otimes E''_\bullet .$$

Corollary 5.6.3. *The addition defined by* $[E_\bullet] \oplus [E'_\bullet] = [E_\bullet \otimes E'_\bullet]$ *makes* $\mathrm{TORS}^n[X; A]$ *an abelian group.*

Proof. $\oplus$ is well defined since $\otimes$ is functorial. The identity element is $0 = [E^s_\bullet]$ where $E^s_\bullet$ is any quasi-split torsor. Associativity, commutativity and the equation $[E_\bullet] \oplus [E^s_\bullet] = [E_\bullet]$ follow immediately from Proposition 5.6.2 . Given $E_\bullet$ define $-E_\bullet$ to be equal to $E_\bullet$ as a simplicial object and $(-\alpha)_m = -(\alpha_m)$. It is straight-forward to verify (again using Proposition 5.6.2) that a quasi-split torsor maps to $E_\bullet \otimes -E_\bullet$ and hence that $[E_\bullet \otimes -E_\bullet] = 0$. □

Theorem 5.6.4. *Given $X' \to X$ and a homomorphism $A \to B$,*

$$\begin{array}{ccc} \mathrm{TORS}^n[X; A] & \longrightarrow & \mathrm{TORS}^n[X; B] \\ \downarrow & & \downarrow \\ \mathrm{TORS}^n[X'; A] & \longrightarrow & \mathrm{TORS}^n[X'; B] \end{array}$$

is a commutative diagram of abelian groups.

Proof. The diagram commutes in sets, by Theorem 4.2. The fact that the maps are homorphisms follows from repeated applications of Theorem 4.2 and the definition of $\otimes$, using the commutativity of

$$\begin{array}{ccc} A \times A & \longrightarrow & B \times B \\ {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} \\ A & \longrightarrow & B \end{array} \qquad \square$$

5.7. Main connected components theorem

From Proposition 2.3.2 we know that every torsor map in $\mathrm{TORS}^1(X; A)$ is an isomorphism. Thus the connected components of 1-torsors are isomorphism classes. The situation is more complicated for $n>1$ since there are n-torsor maps which are not isomorphisms. In particular, the connected component of the canonical quasi-split torsor contains not only all other quasi-split torsors (Lemma 5.5.2) but certain non-quasi-split torsors as well.

Given a homomorphism $f: A \to B$, one may ask about the kernel of

$$\mathrm{TORS}[X;\, f]: \mathrm{TORS}^n[X;\, A] \to \mathrm{TORS}^n[X;\, B].$$

(We will consider this in Chapter 6.) If $\mathrm{TORS}^n[X;\, f][E_\bullet]=0$, we can only infer that $\mathrm{TORS}^n(X;\, f)(E_\bullet)$ is in the connected component containing the quasi-split torsors under B but not that it is quasi-split itself. It is clearly important to know how 'close' $\mathrm{TORS}^n(X;\, f)(E_\bullet)$ is to a quasi-split torsor. That is: how many maps are needed to connect it to a quasi-split torsor? The answer is: one map. We will give a complete proof of this important technical fact.

To begin, let $X_\bullet$ be an arbitrary simplicial object. Then we may define a simplicial object called the 'prisms of $X_\bullet$'.

Definition. Set

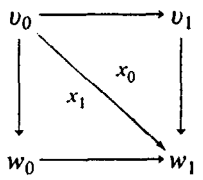
$$\mathrm{Pr}_n X_\bullet = \{(x_0, \ldots, x_n) \in X_{n+1}^{n+1} \mid d_j x_{n-j} = d_j x_{n-j+1} \text{ for } 1 \le j \le n\},$$

$$d_i(x_0, \ldots, x_n) = (d_i x_0, \ldots, d_i x_{n-i-1}, d_{i+1} x_{n-i+1}, d_{i+1} x_{n-i+2}, \ldots, d_{i+1} x_n),$$

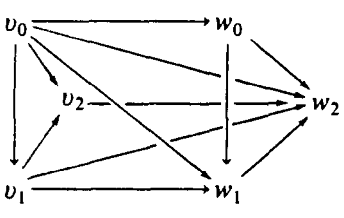
$$s_i(x_0, \ldots, x_n) = (s_i x_0, \ldots, s_i x_{n-i-1}, s_{i+1} x_{n-i-1}, s_{i+1} x_{n-i}, \ldots, s_{i+1} x_n).$$

An element of $\mathrm{Pr}_n X_\bullet$ is an *n-prism*. $\mathrm{Pr}_\bullet X_\bullet$ is a simplicial object.

Remarks and notation. 1. $\mathrm{Pr}_0 X = X_1$, clearly. The word 'prism' is motivated by the following pictures for elements in dimensions 1 and 2. A 1-prism is $(x_0, x_1) \in X_2^2$ such that $d_1 x_0 = d_1 x_1$.

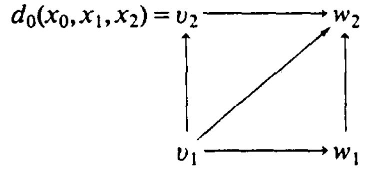


A 2-prism is $(x_0, x_1, x_2) \in X_3^3$ such that $d_1 x_0 = d_1 x_1$ and $d_2 x_1 = d_2 x_2$.



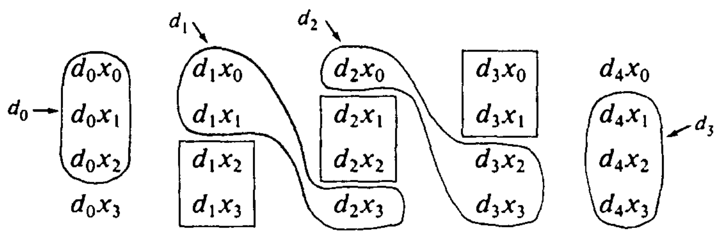
(23)

x_0 is spanned by v_0, v_1, v_2 and w_2. x_1 is spanned by v_0, v_1, w_1 and w_2. x_2 is spanned by v_0, w_0, w_1 and w_2. $d_0(x_0,x_1,x_2) = (d_0x_0, d_0x_1)$. $d_1(x_0,x_1,x_2) = (d_1x_0, d_2x_2)$. $d_2(x_0,x_1,x_2) = (d_3x_1, d_3x_2)$. That is:



etc.

2. For $n>2$ the geometric visualization is impractical but the following matrix-like notation can be very helpful in its stead. Given $(x_0,\dots,x_n) \in \mathrm{Pr}_n X_\bullet \hookrightarrow X_{n+1}^{n+1}$ form the matrix whose i-th row $(0 \le i \le n)$ is $(d_0x_i, d_1x_i, \dots, d_{n+1}x_i)$. The defining equations for $\mathrm{Pr}_n X_\bullet$ and the faces of $(x_0,\dots,x_n)$ appear in easily noted patterns. For example let $(x_0,\dots,x_3) \in \mathrm{Pr}_3 X_\bullet \hookrightarrow X_4^4$.



(24)

The defining equations $d_1x_2 = d_1x_3$, $d_2x_1 = d_2x_2$ and $d_3x_0 = d_3x_1$ are indicated by the boxes in (24). The faces of (x_0,x_1,x_2,x_3) are circled and labeled. The entries d_0x_3 and d_4x_0 will be called the *ends* of $(x_0,\dots,x_3)$. (See the picture in (23).)

3. There are two simplicial maps, the *'end' maps*, $e_\bullet^0 : \mathrm{Pr}_\bullet X_\bullet \to X_\bullet$ and $e_\bullet^1 : \mathrm{Pr}_\bullet X_\bullet \to X_\bullet$ defined by $e_n^0(x_0,\dots,x_n) = d_{n+1}x_0$ and $e_n^1(x_0,\dots,x_n) = d_0x_n$.

Suppose $X_\bullet$ is a groupoid. Note then that a 1-prism is a commutative square and that a 2-prism is a commutative prismatic diagram. Any of the faces (rectangular sides) of a 2-prism is uniquely determined by the other two faces using the groupoid structure of $X_\bullet$. This observation is the gist of the proof of:

Proposition 5.7.1. *If $X_\bullet$ is an n-dimensional hypergroupoid, then so is* $\mathrm{Pr}_\bullet X_\bullet$.

Proof. We must show $\Lambda^m(l)(\mathrm{Pr}_\bullet X_\bullet) = \mathrm{Pr}_l X_\bullet$ for all $l > n$ and $0 \le m \le l$. It will suffice to use the n-dimensional hypergroupoid structure of $X_\bullet$ to show $\Lambda^m(n+1)(\mathrm{Pr}_\bullet X_\bullet) = \mathrm{Pr}_{n+1} X_\bullet$ because a similar argument using the l-dimensional hypergroupoid structure of $X_\bullet$ $(l > n$, see Example 3 of Section 3.1) will verify all the other isomorphisms.

For the rest of the proof, fix m, $0 \le m \le n+1$. Consider

$$(y_0,\dots,y_{n+1}) \in \Delta^\bullet(n+1)(\mathrm{Pr}_\bullet X_\bullet) \hookrightarrow (\mathrm{Pr}_n X_\bullet)^{n+2}$$

where $y_i=(z_{i0},\dots,z_{in})$ is an n-prism. Then $z_{ij}\in X_{n+1}$ and $d_iy_j=d_{j-1}y_i$ for $0\le i<j\le n+1$. The z_{ij}'s form a matrix just as if $\boldsymbol{y}$ consisted of the faces of an $(n+1)$-prism. That is, (M) in (25).

$$(M)\quad\begin{bmatrix}
z_{00} & z_{10} & z_{20} & z_{30} & \cdots & z_{n0} & \square & [\text{end } 0]\\
z_{01} & z_{11} & z_{21} & z_{31} & \cdots & \square & \square & z_{n+1,0}\\
z_{02} & z_{12} & z_{22} & z_{32} & \cdots & \square & z_{n1} & z_{n+1,1}\\
 & \vdots & \vdots & & & z_{n-1,2} & z_{n2} & z_{n+1,2}\\
\vdots & & z_{2,n-2} & \square & & \vdots & \vdots & \vdots\\
 & z_{1,n-1} & \square & & & & & \\
z_{0n} & \square & & z_{2,n-1} & \cdots & z_{n-1,n-1} & z_{n,n-1} & z_{n+1,n-1}\\
[\text{end } 1] & \square & z_{1n} & z_{2n} & \cdots & z_{n-1,n} & z_{nn} & z_{n+1,n}
\end{bmatrix}\tag{25}$$

The empty boxes show where defining equations would equate faces if there were an $(n+1)$-prism whose faces were $y_0,\dots,y_n$. The k-th row of this matrix:

$$(z_{0k}\quad z_{1k}\ \cdots\ z_{n-k,k}\quad \sqcap\quad \sqcup\quad z_{n+2-k,k-1}\ \cdots\ z_{n+1,k-1})$$

consists of $(n+1)$-simplices whose faces match so as to form a hollow $(n+2)$-simplex except for missing faces in slots $n-k+1$ and $n-k+2$.

An element of $\Lambda^m(n+1)(\mathrm{Pr}_\bullet X)$ is like the matrix (M) with $y_m=(z_{m0},\dots,z_{mn})$ missing. We will use the hypergroupoid structure of $X_\bullet$ to fill in this missing element uniquely in terms of the other y_i's.

Since $z_{ij}\in X_{n+1}$, we have $z_{ij}=(t_{ij}^0,\dots,t_{ij}^{n+1})$ with $t_{ij}^k\in X_n$. Thus in order to determine $z_{mj}=(\dots,t_{mj}^k,\dots)$ it will suffice to determine any $n+1$ of its faces; the hypergroupoid structure of $X_\bullet$ will then fill in the remaining face.

Consider the row of (M) in which z_{mj} appears. As we observed, this row forms a hollow $(n+2)$-simplex with two missing faces. Thus the face identities applied in the row containing z_{mj} yield all but two faces of z_{mj}. Those two faces are $d_{n-j}z_{mj}$ and $d_{n-j+1}z_{mj}$. Now since y_m is an n-prism, we have $d_{n-j}z_{mj}=d_{n-j}z_{m,j+1}$ and $d_{n-j+1}z_{mj}=d_{n-j+1}z_{m,j-1}$. Thus the two missing faces of z_{mj} also appear as faces of $z_{m,j+1}$ and $z_{m,j-1}$. We need therefore to find to determine *just one* of the z_{mj}'s (for any value of j); all the others would then be determined by the hypergroupoid structure of $X_\bullet$.

Case $m\le n$. We will find z_{m0}. It will appear in the top row of (M):

$$(z_{00}\quad z_{10}\ \cdots\ z_{m-1,0}\quad (z_{m0})\ \cdots\ z_{n0}\quad \sqcap\quad [\text{end } 0]).$$

$d_i(\text{end } 0)=d_{n+1}z_{i0}$ for $i=0,\dots,n$ and $i\ne m$. $d_{n+1}(\text{end } 0)=d_{n+1}z_{n+1,0}$ (using row 1 of the matrix). The hypergroupoid structure of $X_\bullet$ then determines $d_m(\text{end } 0)=d_{n+1}z_{m0}$ and hence z_{m0}.

Case $m=n+1$. Find $z_{n+1,n}$, using the bottom row of (M) by first finding (end 1) as in the case above. □

Proposition 5.7.2. *If* $X_\bullet$ *is aspherical then so is* $\mathrm{Pr}_\bullet X_\bullet$.

Proof. Let $(y_0,\dots,y_{n+1})\in\Delta^\bullet(n+1)(\mathrm{Pr}_\bullet X_\bullet)$ with $y_i=(z_{i0},\dots,z_{in})\in\mathrm{Pr}_n X$ and form (M) as in the proof of Proposition 5.7.1. We must 'fill in' end 0 and end 1 and all the missing entries in the blank boxes of (M). The rows of such a filled-in matrix would then comprise an element of $\Delta^\bullet(n+2)(X_\bullet)$. By the asphericity of $X_\bullet$ it would then follow that each row consisted of the faces of a $\bar{z}\in X_{n+2}$ so that we'd have $(\bar{z}_0,\dots,\bar{z}_{n+1})\in\mathrm{Pr}_{n+1}X_\bullet$ whose i-th face is y_i. That would complete the proof.

Now all but one face of end 0 and end 1 are already determined by (M). Since $X_\bullet$ is a Kan complex (Corollary 1.7.2) we may choose $(n+1)$-simplices of $X_\bullet$ to fill in for end 0 and end 1. When this is done, each row of (M) is missing only one $(n+1)$-simplex and all of the faces of that simplex are already determined by the face identities. The asphericity of $X_\bullet$ at dimension $n+1$ allows these missing elements to be filled in also. □

Proposition 5.7.3. *If* $X_\bullet\cong\mathrm{COSK}^n(X_\bullet)$ *then* $\mathrm{Pr}_\bullet X_\bullet\cong\mathrm{COSK}^n(\mathrm{Pr}_\bullet X_\bullet)$.

Proof. We will show how to equate $(y_0,\dots,y_{n+1})\in\Delta^\bullet(n+1)(\mathrm{Pr}_\bullet X_\bullet)$ with $(w_0,\dots,w_{n+1})\in\mathrm{Pr}_{n+1}X_\bullet$ using the correspondences

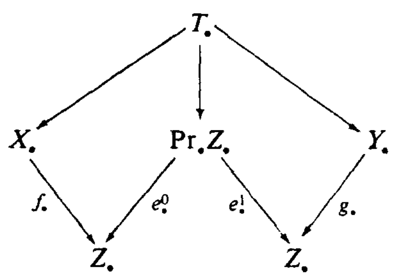
$$(y_0,\dots,y_{n+1})\overset{(1)}{\longleftrightarrow}(M)\overset{(2)}{\longleftrightarrow}(w_0,\dots,w_{n+1}).$$

Correspondence (1) was discussed in Proposition 5.7.1. (Note that $y_i\in X_{n+1}$ is determined by its faces because X_{n+1} is a simplicial kernel). For correspondence (2) consider the matrix (M^*) whose i-th row is $(d_0w_i,\dots,d_{n+2}w_i)$. If end 0, end 1 and d_jw_{n-j} and d_jw_{n-j+1} $(1\le j\le n)$ are deleted from (M^*) then one obtains an '(M)' matrix. This process is reversible because the faces of the deleted entries can be recovered from the undeleted entries using the simplicial identities. □

Lemma 5.7.4. *Given simplicial maps*

$$X_\bullet\xrightarrow{f_\bullet}Z_\bullet\xleftarrow{g_\bullet}Y_\bullet,$$

let $T_\bullet$ *be the limit indicated in the diagram*



Then: (i) *If* $X_\bullet \cong \mathrm{COSK}^n(X_\bullet)$, $Y_\bullet \cong \mathrm{COSK}^n(Y_\bullet)$ *and* $Z_\bullet \cong \mathrm{COSK}^n(Z_\bullet)$, *then* $T_\bullet \cong \mathrm{COSK}^n(T_\bullet)$.

(ii) *If* $X_\bullet$ *and* $Y_\bullet$ *and* $Z_\bullet$ *are aspherical, then so is* $T_\bullet$.

Proof. (i) This follows from the commutativity of limits.

(ii) An element of T_n is $(x,(z_0,\dots,z_n),y) \in X_n \times \mathrm{Pr}_n Z \times Y_n$ such that $f_n x = e_n^0(z_0,\dots,z_n) = d_{n+1}z_0$ and $g_n y = e_n^1(z_0,\dots,z_n) = d_0 z_n$. An element of $\Delta^\bullet(n+1)(T_\bullet)$ is thus

$$\begin{bmatrix} x_0 & (z_0,\dots,z_n)_0 & y_0 \\ \vdots & \vdots & \vdots \\ x_{n+1} & (z_0,\dots,z_n)_{n+1} & y_{n+1} \end{bmatrix}$$

with $(x_i,(z_0,\dots,z_n)_i,y_i) \in T_n$ and $(z_0,\dots,z_n)_i$ abbreviating $(z_{i0},\dots,z_{in})$. Also,

$$\begin{aligned} d_i(x_j,(z_0,\dots,z_n)_j,y_j) &= (d_i x_j, d_i(z_0,\dots,z_n)_j, d_i y_j) \\ &= d_{j-1}(x_i,(z_0,\dots,z_n)_i,y_i) \quad \text{for } i<j. \end{aligned}$$

Clearly, $(x_0,\dots,x_{n+1}) \in \Delta^\bullet(n+1)(X_\bullet)$ and $(y_0,\dots,y_{n+1}) \in \Delta^\bullet(n+1)(Y_\bullet)$. By hypothesis there exist $\bar{x} \in X_{n+1}$ and $\bar{y} \in Y_{n+1}$ such that $d_i\bar{x} = x_i$ and $d_i\bar{y} = y_i$ for each i. We then need to find $(\bar{z}_0,\dots,\bar{z}_{n+1}) \in \mathrm{Pr}_{n+1}Z_\bullet$ such that

$$d_i(\bar{z}_0,\dots,\bar{z}_{n+1}) = (z_0,\dots,z_n)_i = (z_{i0},\dots,z_{in}).$$

This can be done by Proposition 5.7.2 so that z_0 and z_{n+1} satisfy $d_{n+2}\bar{z}_0 = f_{n+1}\bar{x}$ and $d_0\bar{z}_{n+1} = g_{n+1}\bar{y}$ thus yielding $(\bar{x},(\bar{z}_0,\dots,\bar{z}_{n+1}),\bar{y}) \in T_{n+1}$. □

Theorem 5.7.5. *Let*

$$E_\bullet \xrightarrow{\varphi_\bullet} E'_\bullet \xleftarrow{\psi_\bullet} E''_\bullet$$

be torsor maps in $\mathrm{TORS}^n(X;A)$. *Then there is a torsor* $\tilde{E}_\bullet \in \mathrm{TORS}^n(X;A)$ *and maps* $E_\bullet \leftarrow \tilde{E}_\bullet \rightarrow E''_\bullet$.

Proof. Using $\varphi_\bullet$ and $\psi_\bullet$ as in Lemma 5.7.4, form $T_\bullet$, set $\tilde{E}_\bullet = T_\bullet$ and take $E_\bullet \leftarrow \tilde{E}_\bullet \rightarrow E''_\bullet$ to be the projections. We have $\tilde{E}_\bullet \cong \mathrm{COSK}^{n-1}(\tilde{E}_\bullet)$ and aspherical by Lemma 5.7.4. We must show $\tilde{E}_\bullet$ is a torsor and that the projections are torsor maps. An element of $\Delta^\bullet(n)(\tilde{E}_\bullet)$ is

$$\tilde{y} = \begin{bmatrix} y_0 & (y'_0,\dots,y'_{n-1})_0 & y''_0 \\ \vdots & \vdots & \vdots \\ y_i & (y'_0,\dots,y'_{n-1})_i & y''_i \\ \vdots & \vdots & \vdots \\ y_n & (y'_0,\dots,y'_{n-1})_n & y''_n \end{bmatrix}$$

We must show that the i-th row is uniquely determined by the other rows and by the element $a \in A$ to which $\tilde{y}$ is sent by the composite $\tilde{E}_n \rightarrow E_n \rightarrow A$. This will simultaneously verify $\tilde{E}_n \cong \Lambda^i(n)(E_\bullet) \times A$ and that $\tilde{E}_\bullet \rightarrow E_\bullet$ and $\tilde{E}_\bullet \rightarrow E''_\bullet$ are torsor maps. First, y_i and y''_i are uniquely determined by the torsor structures of $E_\bullet$ and $E''_\bullet$. As for $(y'_0, \ldots, y'_{n-1})_i = (y'_{i0}, \ldots, y'_{i,n-1})$, observe that we must have $\varphi_{n-1} y_i = d_n y'_{i0}$ and $\psi_{n-1} y''_i = d_0 y'_{i,n-1}$ and that

$$(\ldots, (y'_0, \ldots, y'_{n-1})_i, \ldots) \in \Delta^\bullet(n)(\mathrm{Pr}_\bullet E'_\bullet).$$

These conditions determine all but one face of y'_{i0} and $y'_{i,n-1}$ (use matrix (M) for this). The $(n-1)$-dimensional hypergroupoid structure of the fiber of $E'_\bullet$ then determines y'_{i0} and $y'_{i,n-1}$ uniquely. Similarly, we can successively determine y'_{i1} and $y'_{i,n-2}$, y'_{i2} and $y'_{i,n-3}$, etc. □

Corollary 5.7.6. *If $E_\bullet$ and $E'_\bullet$ are in the same connected component of* $\mathrm{TORS}^n(X; A)$, *then there is a torsor $\tilde{E}_\bullet$ and maps $E_\bullet \leftarrow \tilde{E}_\bullet \rightarrow E'_\bullet$.*

Proof. A sequence of torsor maps connecting $E_\bullet$ and $E'_\bullet$ looks like

$$E_\bullet \rightarrow T^1 \leftarrow T^2 \rightarrow T^3 \leftarrow T^4 \cdots T^m \rightarrow T^{m+1} \leftarrow E'_\bullet.$$

Each 'corner'

$$T^k \rightarrow T^{k+1} \leftarrow T^{k+2}$$

can be replaced by

$$T^k \leftarrow \tilde{T}^k \rightarrow T^{k+2}$$

Repeating this replacement process eventually yields $E_\bullet \leftarrow \tilde{E}_\bullet \rightarrow E'_\bullet$. □

Corollary 5.7.7. *If* $[E_\bullet] = 0 \in \mathrm{TORS}^n[X; A]$, *then there is a torsor map $E^s_\bullet \rightarrow E_\bullet$ where E^s is quasi-split.*

Proof. We have $E_\bullet \leftarrow \tilde{E}_\bullet \rightarrow E^\#_\bullet$ where $E^\#_\bullet$ is the canonical quasi-split torsor. $\tilde{E}_\bullet$ must then be quasi-split, by Corollary 3.8.2. □

6. The long exact sequence of cohomology

In this chapter we will show how a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian group objects in the exact category $\mathscr{C}$ determines a homomorphism

$$\delta_n : \mathrm{TORS}^n[X;\, C] \longrightarrow \mathrm{TORS}^{n+1}[X;\, A]$$

for each $n>0$ and also show that the long sequence of cohomology groups:

$$\cdots \longrightarrow \mathrm{TORS}^n[X;\, A] \longrightarrow \mathrm{TORS}^n[X;\, B] \longrightarrow \mathrm{TORS}^n[X;\, C]$$
$$\xrightarrow{\delta_n} \mathrm{TORS}^{n+1}[X;\, A] \longrightarrow \cdots$$

is exact. The proofs involve the material on fibers of torsors, torsors under hypergroupoids and connected components of $\mathrm{TORS}^n(X;\, A)$ developed in earlier chapters.

6.1. *Preliminary facts*

Let $f: A \to B$ be any homomorphism of abelian group objects. For future reference in this chapter, let us review the functor $\mathrm{TORS}^n(X;\, f): \mathrm{TORS}^n(X;\, A) \to \mathrm{TORS}^n(X;\, B)$. If $\mathrm{TORS}^n(X;\, f)(E_\bullet) = E'_\bullet$, then on the attached 1-torsor level we have $\mathrm{TORS}^1(\Delta^\bullet(n-1)(E_\bullet);\, f)(E_{\bullet,1}) = E'_{\bullet,1}$ and the diagram

$$\begin{array}{ccc}
E_{n-1} \times A \times B & \longrightarrow & E_{n-1} \times A \\
D_0 \downarrow\downarrow D_1 & & d_0 \downarrow\downarrow d_1 \\
E_{n-1} \times B & \xrightarrow{\ \mathrm{pr}\ } & E_{n-1} \\
q \downarrow & & \downarrow p \\
E'_{n-1} & \xrightarrow{\ p'\ } & \Delta^\bullet(n-1)(E_\bullet)
\end{array}$$

with $D_0(y,a,b) = (y,b)$, $D_1(y,a,b) = (ya, -fa+b)$, $q(y,b)b' = q(y, b+b')$ and $p'q(y,b) = py$.

Lemma 6.1.1. *Given $f: A \to B$ and $E_\bullet \in \mathrm{TORS}^n(X;\, A)$, let*

$$\phi_\bullet : E'_\bullet \to \mathrm{TORS}^n(X;\, f)(E_\bullet) = \bar{E}_\bullet$$

be an n-torsor map in $\mathrm{TORS}^n(X;\, B)$. Then there exists a torsor map $\varphi_\bullet : E'_\bullet \to E_\bullet$ in $\mathrm{TORS}^n(X;\, A)$ such that $\mathrm{TORS}^n(X;\, f)(\varphi_\bullet) = \phi_\bullet$.

Proof. By Proposition 3.8.1, the map $\mathrm{Tr}^{n-2}(\phi_\bullet) : \mathrm{Tr}^{n-2}(E'_\bullet) \to \mathrm{Tr}^{n-2}(\bar{E}_\bullet) = \mathrm{Tr}^{n-2}(E_\bullet)$ extends to a torsor map $\varphi_\bullet : E'_\bullet \to E_\bullet$ in $\mathrm{TORS}^n(X;\, A)$. Suppose $\mathrm{TORS}^n(X;\, f)(\varphi_\bullet) = \varphi''_\bullet : E''_\bullet \to \bar{E}_\bullet$. It will suffice by Corollary 3.8.2 to show that $\varphi''_\bullet$ factors through $\phi_\bullet$.

$$\begin{array}{ccc}
E''_\bullet & \longrightarrow & \bar{E}_\bullet \\
 & \searrow\!\!\cdot\ \ \nearrow & \\
 & E'_\bullet &
\end{array}$$

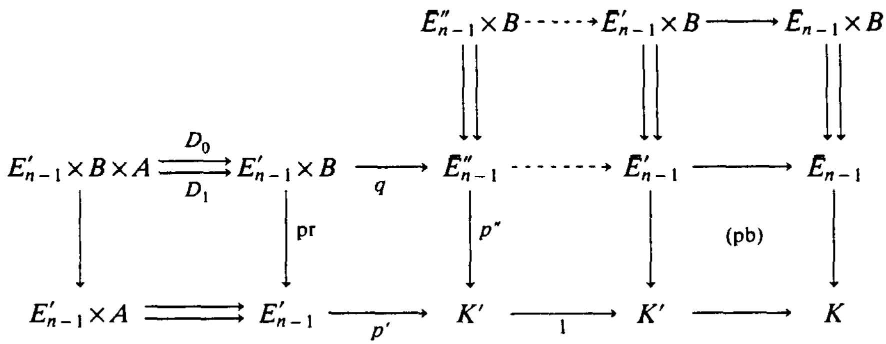
Consider diagram (26). The dotted arrows exist because

$$E'_{\bullet,1} \xrightarrow{\Phi_{\bullet\bullet 1}} E_{\bullet,1}$$

is a pullback. Thus

$$\begin{array}{ccc} E''_{n-1} & \longrightarrow & E'_{n-1} \\ \downarrow & & \downarrow \\ K' & \xrightarrow[1]{} & K' \end{array}$$

is a pullback. This shows in fact that $\mathrm{TORS}^n(X;\, f)(E'_\bullet) = E''_\bullet \cong E'_\bullet$.



(26)

($K' = \Delta^\bullet(n-1)(E'_\bullet)$ and $K = \Delta^\bullet(n-1)(E_\bullet)$).

6.2. *The connecting homomorphism*

Fix a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian group objects in $\mathscr{C}$. This means that f is monic, g is epic and $gb = 0$ iff $b = fa$. Equivalently, f and g determine a 1-torsor

$$B \times A \rightrightarrows B \xrightarrow[g]{} C$$

where the action of A on B is defined by $ba = b + fa$.

Now the n-dimensional hypergroupoids $K(A,n)$, $K(B,n)$ and $K(C,n)$ are abelian group objects in $\mathrm{Hypgpd}_n(\mathscr{C})$ and thus f and g also determine a 1-torsor

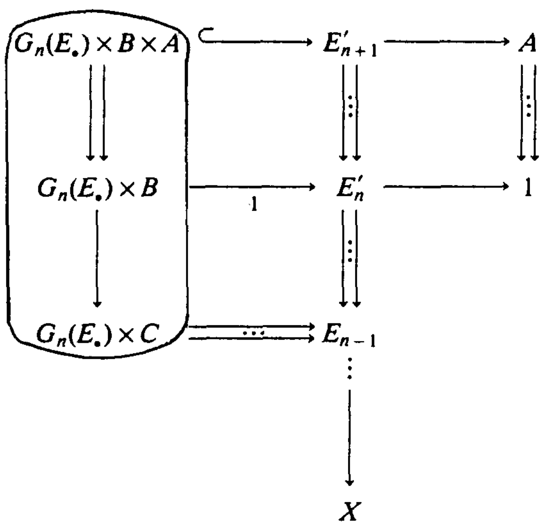
$$K(B,n) \times K(A,n) \rightrightarrows K(B,n) \longrightarrow K(C,n)$$

in that category. By Theorem 5.3.1 there is a corresponding $(n+1)$-torsor under A over 1.

Suppose now that $\alpha_\bullet : E_\bullet \to K(C,n)$ is an n-torsor. Let $G_\bullet(E_\bullet)$ be the fiber of $E_\bullet$. Recall (Section 5.2) that $G_\bullet(E_\bullet)$ is an $(n-1)$-dimensional hypergroupoid (and thus also an n-dimensional hypergroupoid). We have $y \in G_n(E_\bullet) \hookrightarrow E_n$ iff $\alpha_n y = 0$. Now form the pullback 1-torsor in $\text{Hypgpd}_n(\mathscr{C})$:

$$\begin{array}{ccccc} G_\bullet(E_\bullet)\times K(B,n)\times K(A,n) & \rightrightarrows & G_\bullet(E_\bullet)\times K(B,n) & \longrightarrow & G_\bullet(E_\bullet)\times K(C,n) \\ \downarrow{\scriptstyle \text{pr}} & & \downarrow{\scriptstyle \text{pr}} & & \downarrow{\scriptstyle \text{pr}} \\ K(B,n)\times K(A,n) & \rightrightarrows & K(B,n) & \longrightarrow & K(C,n) \end{array}$$

By Theorem 5.3.1 this yields an $(n+1)$-torsor $\alpha'_\bullet : E'_\bullet \to K(A, n+1)$ over X.



(27)

The attached 1-torsor of $E'_\bullet$ is circled in (27).

The correspondence $\delta_n : \text{TORS}^n(X;\ C) \to \text{TORS}^{n+1}(X;\ A)$ defined by $\delta_n(\alpha_\bullet) = \alpha'_\bullet$ is obviously functorial.

Let us determine α'_{n+1} explicitly. Suppose $(y,b) \in G_n(E_\bullet)\times B = E'_n$. Using the isomorphism $G_n(E_\bullet) \cong \Lambda^n(n)(E_\bullet)$ we then have $d_i(y,b) = y_i$ for $0 \le i \le n-1$ and $d_n(y,b) = y_n g(b)$. Since E'_{n+1} is a simplicial kernel, an element of E'_{n-1} is

$$((y_0,b_0),\ldots,(y_{n+1},b_{n+1}))$$

where $d_i(y_j,b_j) = d_{j-1}(y_i,b_i)$ for all $i<j$, as usual. We must define

$$\alpha'_{n+1}(\ldots,(y_i,b_i),\ldots).$$

Following the proof of Theorem 5.3.1, consider

$$((y_0,b_0),\ldots,(y_n,b_n),(y'_{n+1},b'_{n+1})) \in G_{n+1}(E_\bullet)\times K(B,n)_{n+1}.$$

Here, $d_i(y'_{n+1}) = d_n y_i = y_{in}$ and $\text{A.S.}(b_0,\ldots,b_n,b'_{n+1}) = 0$. Then since $d_i(y_{n+1},b_{n+1}) =$

$d_i(y'_{n+1}, b'_{n+1})$ for each $i=0,\dots,n$, it follows (see the attached 1-torsor of $E'_\bullet$) that

$$(y_{n+1}, b_{n+1}) = (y'_{n+1}, b'_{n+1})a = (y'_{n+1}, b'_{n+1}+fa)$$

for some unique $a \in A$. Then $\alpha'_{n+1}(\dots,(y_i,b_i),\dots)=a$. Now $b'_{n+1}+fa=b_{n+1}$ and thus

$$fa = b_{n+1} - b'_{n+1} = b_{n+1} - \text{A.S.}(b_0,\dots,b_n) = \text{A.S.}(b_0,\dots,b_{n+1}).$$

Proposition 6.2.1. *Given a commutative diagram of short exact sequences*

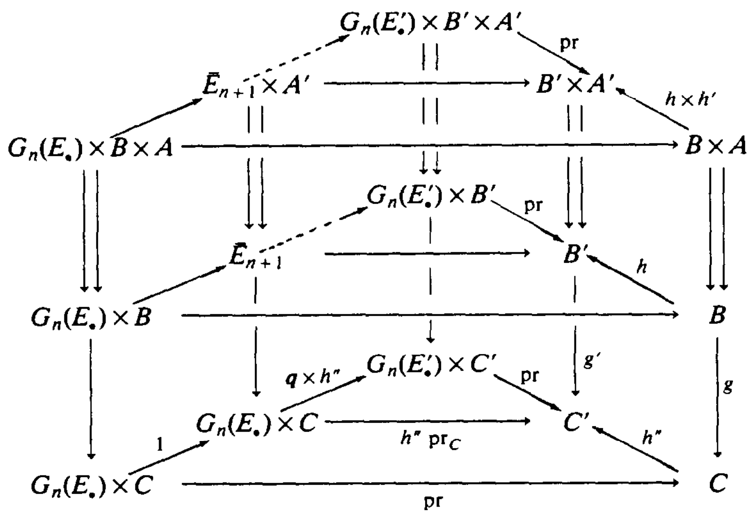
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h''} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0 \end{array}$$

then

$$\begin{array}{ccc} \text{TORS}^n[X;\,C] & \xrightarrow{\delta_n} & \text{TORS}^{n+1}[X;\,A] \\ \downarrow & & \downarrow \\ \text{TORS}^n[X;\,C'] & \xrightarrow[\delta_n]{} & \text{TORS}^{n+1}[X;\,A'] \end{array}$$

commutes.

Proof. Let $E_\bullet$ be in $\text{TORS}^n(X;\,C)$ with fiber $G_\bullet(E_\bullet)$ and consider diagram (28). The left-most column is the attached 1-torsor of $\delta_n(E_\bullet)$. The column involving $\bar{E}_{n+1}$ is the pullback torsor of $B'\times A' \rightrightarrows B' \to C'$ along $h''\,\text{pr}_C : G_n(E_\bullet)\times C \to C \to C'$ and is the attached 1-torsor of $\bar{E}_\bullet \in \text{TORS}^{n+1}(X;\,A')$. The induced maps between these torsors is readily checked to be equivariant.



(28)

It follows from Proposition 4.3 and Lemma 2.4.3 that the map $\delta_n(E_\bullet)\to\bar{E}_\bullet$ factors as

$$\delta_n(E_\bullet)\longrightarrow \mathrm{TORS}^{n+1}(X;\ h')\delta_n(E_\bullet)\xrightarrow{\ \approx\ }\bar{E}_\bullet$$

where the second map is an $(n+1)$-torsor map. It is, in fact, an isomorphism because it consists of identity maps in dimensions $\leq n-1$. The center column containing $G_n(E'_\bullet)\times B'$ is the attached 1-torsor of $\delta_n(E'_\bullet)=\delta_n(\mathrm{TORS}^n(X;\ h'')(E_\bullet))$. The map $G_n(E_\bullet)\times C\to G_n(E'_\bullet)\times C'$ sends (y,c) to $(qy,\ h''c)$ where $qy=(q(y_0,0),\dots,q(y_n,0))$ and q is as in Section 6.1. One has $qy\in G_n(E'_\bullet)$ because f is monic. The dotted $(n+1)$-torsor map can then be defined sending $(y,c,b')\in\bar{E}_{n+1}$ to $(qy,b')\in G_n(E'_\bullet)\times B'$. We thus have

$$\mathrm{TORS}^{n+1}(X;\ h')\delta_n(E_\bullet)\longrightarrow \delta_n\mathrm{TORS}^n(X;\ h'')(E_\bullet)$$

and therefore $\mathrm{TORS}^{n+1}[X;\ f]\delta_n=\delta_n\mathrm{TORS}^n[X;\ h'']$. □

Proposition 6.2.2. *Given the short exact sequence*

$$0\longrightarrow A\xrightarrow{\ f\ }B\xrightarrow{\ g\ }C\longrightarrow 0$$

and a map $Y\to X$, then

$$\begin{array}{ccc}\mathrm{TORS}^n[X;\ C] & \longrightarrow & \mathrm{TORS}^{n+1}[X;\ A]\\ \Big\downarrow & & \Big\downarrow\\ \mathrm{TORS}^n[Y;\ C] & \longrightarrow & \mathrm{TORS}^{n+1}[Y;\ A]\end{array}$$

commutes.

Proof. Given $E_\bullet\in\mathrm{TORS}^n(X;\ C)$ let $E'_\bullet$ be the pullback torsor of $E_\bullet$ along $Y\to X$. Then the projection $E'_\bullet\to E_\bullet$ is a C-equivariant map which determines an A-equivariant map $\delta_n(E'_\bullet)\to\delta_n(E_\bullet)$. It is obvious from the definition of δ_n that $\delta_n(E'_\bullet)$ is the pullback torsor of $\delta_n(E_\bullet)$, and this proves the proposition. □

Corollary 6.2.3. $\delta_n:\mathrm{TORS}^n[X;\ C]\to\mathrm{TORS}^{n+1}[X;\ A]$ *is a homomorphism.*

Proof. Consider the map of short exact sequences

$$\begin{array}{ccccccccc}0 & \longrightarrow & A\times A & \longrightarrow & B\times B & \longrightarrow & C\times C & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle +} & & \Big\downarrow{\scriptstyle +} & & \Big\downarrow{\scriptstyle +} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0\end{array}$$

The claim follows by applying Propositions 6.2.1 and 6.2.2 to the definitions of ⊗ and ⊕. □

The case of $\delta_0: \mathscr{C}(X, C) \to \mathrm{TORS}^1(X; A)$ goes as follows. Given $t: X \to C$, $\delta_0(t)$ is the 1-torsor formed by pulling back along t as shown:

$$\begin{array}{ccc}
E_0 \times A & \longrightarrow & B \times A \\
\downarrow\downarrow & & \downarrow\downarrow \\
E_0 & \longrightarrow & B \\
\downarrow & \text{(pb)} & \downarrow \\
X & \xrightarrow[t]{} & C
\end{array}$$

The theorems corresponding to 6.2.1–6.2.3 are easily established and left to the reader.

6.3. *The exactness of the long sequence*

We are now ready to show that the long sequence of cohomology determined by the short exact sequence $(f, g): 0 \to A \to B \to C \to 0$ is exact. For readability we will use 'f^*' to denote both $\mathrm{TORS}^n[-; f]$ and $\mathrm{TORS}^n(-; f)$.

Theorem 6.3.1.

$$\mathrm{TORS}^n[X; C] \xrightarrow{\delta_n} \mathrm{TORS}^{n+1}[X; A] \xrightarrow{f^*} \mathrm{TORS}^{n+1}[X; B]$$

is exact.

Proof. Let $E_\bullet \in \mathrm{TORS}^{n+1}[X; A]$ and set $E'_\bullet = f^* E_\bullet$. First we will show $f^* \delta_n = 0$. Suppose $n = 0$, $t: X \to C$, and $E_\bullet = \delta_0 t$.

$$\begin{array}{ccccc}
E_0 \times A \times B & \longrightarrow & E_0 \times A & \longrightarrow & B \times A \\
D_0 \downarrow\downarrow D_1 & & \downarrow\downarrow & & \downarrow\downarrow \\
E_0 \times B & \longrightarrow & E_0 & \xrightarrow{\ \mathrm{pr}\ } & B \\
q_0 \downarrow & & \downarrow p & & \downarrow g \\
E'_0 & \xrightarrow[p']{} & X & \xrightarrow[t]{} & C
\end{array}$$

Recall that p is the pullback of g along t. Define $h_\bullet: E_\bullet \times K_\#(B, 1) \to K_\#(B, 1)$ by setting $h_1(y, b) = b + \mathrm{pr}(y)$. Then $h_1 D_0 = h_1 D_1$ since

$$\begin{aligned}
h_1 D_1(y, a, b) &= h_1(ya, -fa + b) = -fa + b + \mathrm{pr}(ya) \\
&= -fa + b + fa + \mathrm{pr}(y) = b + \mathrm{pr}(y) = h_1 D_0(y, a, b).
\end{aligned}$$

Thus $h_\bullet$ factors through $E'_\bullet$ yielding

$$E'_\bullet \xrightarrow{w_\bullet} K_\#(B,1) \longrightarrow K(B,1).$$

This proves $f^*\delta_0 = 0$.

In dimensions $n > 0$ the same argument shows that the attached 1-torsor of $f^*\delta_n E_\bullet$ is split and hence that $f^*\delta_n[E_\bullet] = 0$.

Next we must show $\ker(f^*) \subsetneq \operatorname{im}(\delta_n)$. Assume $f^*[E_\bullet] = 0$. By Corollary 5.7.7 we then have (quasi-split torsor)$\to f^*E_\bullet$ and by Lemma 6.1.1 we know there is a torsor in the same connected component as $E_\bullet$ which f^* maps to the quasi-split one. Let us then suppose that we have chosen a representative of $[E_\bullet]$, $E_\bullet$ itself without loss of generality, such that f^* sends this representative to a quasi-split torsor in $\mathrm{TORS}^{n+1}(X;B)$.

Consider the case $n = 0$. There is a map $v: E_0 \to B$ defined by $vy = w_0 q_0(y, 0)$. Also,

$$v(ya) = w_0 q_0(ya, 0) = w_0(q_0(y,0)fa) = w_0 q_0(y,0) + fa = vy + fa.$$

Hence we have

$$\begin{array}{ccc} E_0 \times A & \xrightarrow{v \times 1} & B \times A \\ \downarrow\downarrow & & \downarrow\downarrow \\ E_0 & \xrightarrow{v} & B \\ \downarrow & & \downarrow \\ X & \dashrightarrow_{t} & C \end{array}$$

which induces $t: X \to C$ in the quotient. Obviously $E_\bullet = \delta_0 t$ and so we have shown $\ker(f^*) \subsetneq \operatorname{im}(\delta_0)$.

Now consider the case $n > 0$. Regard $E_\bullet \in \mathrm{TORS}^{n+1}(X;A)$ as a 1-torsor

$$\mathrm{Fib}(E_\bullet) \times K(A,n) \rightrightarrows \mathrm{Fib}(E_\bullet) \longrightarrow \mathrm{COSK}^{n-1}(E_\bullet)$$

in the category $\mathrm{Hypgpd}_n(\mathscr{C})$ according to Theorem 5.3.1. As we just saw in the $n = 0$ case, $f^*E_\bullet$ being split (in $\mathrm{Hypgpd}_n(\mathscr{C})$) implies there is a map $t_\bullet: \mathrm{COSK}^{n-1}(E_\bullet) \to K(C,n)$ such that $E_\bullet$ is the pullback torsor of

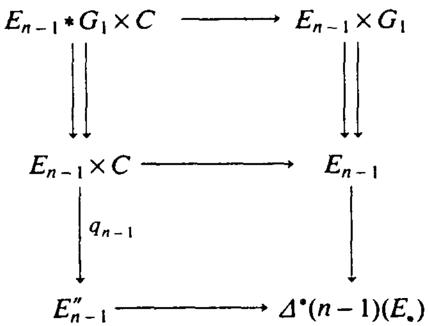
$$K(B,n) \times K(A,n) \rightrightarrows K(B,n) \longrightarrow K(C,n)$$

along $t_\bullet$. At dimension n the square

$$\begin{array}{ccc} E_n & \xrightarrow{v} & B \\ \downarrow & & \downarrow g \\ \Delta^\bullet(n)(E_\bullet) & \xrightarrow[t_n]{} & C \end{array}$$

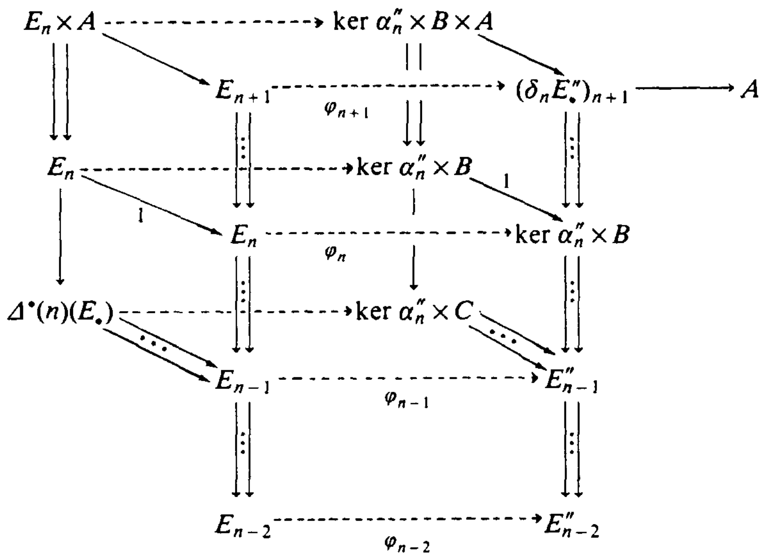
is a pullback.

The identity map on the n-dimensional hypergroupoid $\text{COSK}^{n-1}(E_\bullet)$ makes this hypergroupoid a torsor under itself. This special torsor is sent by the functor $\text{TORS}(X;\, t_\bullet)$ to an n-torsor $E''_\bullet \in \text{TORS}^n(X;\, C)$. We will now show that $\delta_n([E''_\bullet]) = [E_\bullet]$ by finding a torsor map $E_\bullet \to \delta_n E''_\bullet$. From the definition of the functor $\text{TORS}(X;\, t_\bullet)$ (see Theorem 4.1, diagram (17)) we are concerned with the diagram in (29) where $G_\bullet$ is the associated groupoid of $\text{COSK}^{n-1}(E_\bullet)$.



(29)

We need to define the dotted arrows in (30) representing $\varphi_\bullet : E_\bullet \to \delta_n E''_\bullet$.



(30)

($\alpha''_\bullet : E''_\bullet \to K(C, n)$ is the action for $E''_\bullet$).

In this diagram set $\varphi_m = 1$ for $m = 0, \dots, n-2$ since $E''_m = E_m$ for such m. Set $\varphi_{n-1} y = q_{n-1}(y, 0)$. Recall the definition in Section 6.2 of the face maps

$$d_i : \ker \alpha''_n \times B \to E''_{n-1}.$$

For $i < n$, $d_i(\bar{y}_0, \dots, \bar{y}_n, b) = \bar{y}_i$ and $d_n(\bar{y}_0, \dots, \bar{y}_n, b) = \bar{y}_n g(b)$. This forces the definition of φ_n to be:

$$\varphi_n y = (q_{n-1}(d_0 y, 0), \dots, q_{n-1}(d_{n-1} y, 0), q_{n-1}(d_n y, -g v y), v y),$$

observing that $q_{n-1}(d_n y, -gvy) = q_{n-1}(d_n y, 0)(-gvy)$. This determines $\varphi_\bullet$ as a simplicial map since everything in higher dimensions consists of simplicial kernels. In order to see that $\varphi_\bullet$ is a torsor map, consider diagram (31).

$$\begin{array}{ccccc} E_n & \xrightarrow{\varphi_n} & \ker\alpha''_n\times B & \xrightarrow{\mathrm{pr}_B} & B \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ \Delta^\bullet(n)(E_\bullet) & \xrightarrow[\varphi_{n-1}]{} & \ker\alpha''_n\times C & \xrightarrow[\mathrm{pr}_C]{} & C \end{array} \tag{31}$$

Note that $\mathrm{pr}_B\,\varphi_n = v$. We also have $\mathrm{pr}_C\,\varphi_{n-1} = t_n$. To see this, recall that $\ker\alpha''_n \times C = E''_n$. Thus

$$\begin{aligned}\mathrm{pr}_C\,\varphi_{n-1}\boldsymbol{y} &= \alpha''_n\varphi_{n-1}\boldsymbol{y} = \alpha''_n(\ldots, q_{n-1}(y_i, 0), \ldots)\\ &= \alpha''_n q_n(\ldots, (y_i, 0), \ldots), \ldots) = \zeta(\ldots, (y_i, 0), \ldots)\end{aligned}$$

where 'ζ' is the map defined in the proof of Theorem 4.1 and which appears in diagram (18). Following the definition of 'ζ' as it applies in this particular case,

$$\zeta((y_0, c_0), \ldots, (y_n, c_n)) = \mathrm{A.S.}(c_0, \ldots, c_n) + t_n(\boldsymbol{y}).$$

Thus $\zeta(\ldots, (y_i, 0), \ldots) = t_n(\boldsymbol{y})$. Since v is the pullback of g along t_n the outside square of diagram (31) is a pullback. The right-hand square of that diagram is also a pullback and therefore the left-hand square is a pullback. By Corollary 3.8.2, $\varphi_\bullet$ must then be a torsor map. This completes the proof that $\ker(f^*) \subseteq \mathrm{im}(\delta_n)$. □

Theorem 6.3.2.

$$\mathrm{TORS}^n[X; B] \xrightarrow{g^*} \mathrm{TORS}^n[X; C] \xrightarrow{\delta_n} \mathrm{TORS}^{n+1}[X; A]$$

is exact.

Proof. First suppose $n = 0$. We must show

$$\mathscr{C}(X; B) \xrightarrow{g^*} \mathscr{C}(X, C) \xrightarrow{\delta_0} \mathrm{TORS}^1[X; A]$$

is exact. To see that $\delta_0 g^* = 0$ consider the diagram

$$\begin{array}{ccccc} (\delta_0 gu)_0 & \longrightarrow & B * B & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[u]{} & B & \xrightarrow[g]{} & C \end{array}$$

where both squares are pullbacks. Since $B * B \to B$ is split by the diagonal, $\delta_0 gu$ is a split torsor.

Next, if $E_\bullet = \delta_0(t : X \to C)$ is split we have

$$\begin{array}{ccc} E_0 & \xrightarrow{\ \mathrm{pr}\ } & B \\ {\scriptstyle s}\uparrow\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\ t\ ]{} & C \end{array}$$

and so $t = g\,\mathrm{pr}\,s = g^*(\mathrm{pr}\,s)$. Thus $\ker(\delta_0) = \mathrm{im}(g^*)$.

Now suppose $n > 0$. First we will show $\delta_n g^* = 0$. Let $\alpha_\bullet : E_\bullet \to K(B, n)$ be an n-torsor under B. Consider diagram (32) in the category $\mathrm{Hypgpd}_n(\mathscr{C})$.

$$\begin{array}{ccccc} K_\#(A,n) \times \mathrm{COSK}^{n-1}(E_\bullet) \times K(A,n)^2 & \dashrightarrow & G_\bullet(g^*E_\bullet) \times K(A,n) & \longrightarrow & K(B,n) \times K(A,n) \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ K_\#(A,n) \times \mathrm{COSK}^{n-1}(E_\bullet) \times K(A,n) & \overset{}{\underset{\psi_\bullet}{\dashrightarrow}} & G_\bullet(g^*E_\bullet) & \longrightarrow & K(B,n) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g_\bullet} \\ K_\#(A,n) \times \mathrm{COSK}^{n-1}(E_\bullet) & \underset{\varphi_\bullet}{\dashrightarrow} & g^*E_\bullet & \xrightarrow[g^*\alpha_\bullet]{} & K(C,n) \end{array} \tag{32}$$

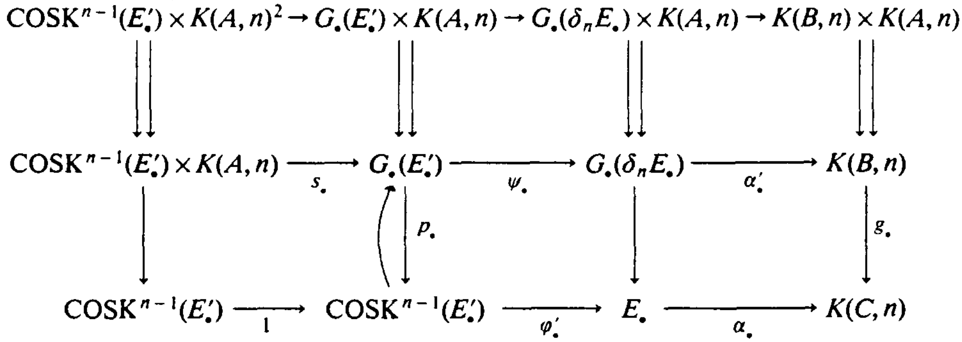
$G_\bullet(g^*E_\bullet)$ is the fibre of $E_\bullet$ and the middle column is the 1-torsor corresponding to $\delta_n g^* E_\bullet$ in the category $\mathrm{Hypgpd}_n(\mathscr{C})$. The right side of the diagram is a pullback of 1-torsors. The left-most column is the product of $\mathrm{COSK}^{n-1}(E_\bullet)$ with the canonical split 1-torsor under $K(A, n)$, again in the category $\mathrm{Hypgpd}_n(\mathscr{C})$. The existence of the dotted maps to produce a pullback diagram of 1-torsors will show that $[\delta_n g^* E_\bullet] = 0$. In dimension n, an element of $K_\#(A, n) \times \mathrm{COSK}^{n-1}(E_\bullet)$ is $(a, y, b) \in A \times \ker(\alpha_n) \times B$. An element of $(g^*E_\bullet)_n$ is $(\bar{y}_0, \ldots, \bar{y}_n, c)$ where $(\bar{y}_0, \ldots, \bar{y}_n) \in \ker(g^*\alpha_\bullet)_n$ and $(g^*\alpha_\bullet)_n(\bar{y}_0, \ldots, \bar{y}_n, c) = c$. Recall, from the definition of g^*, the diagram

$$\begin{array}{ccc} E_n \times C & \longrightarrow & E_{n-1} \\ \downarrow{\scriptstyle q_{n-1}} & & \downarrow \\ (g^*E_\bullet)_{n-1} & \longrightarrow & \Delta^\bullet(n-2)(E_\bullet) \end{array}$$

Define $\varphi_n(a, y, b) = (q_{n-1}(y_0, 0), \ldots, q_{n-1}(y_n, 0), gb)$. Thus $(g^*\alpha_\bullet)_n \varphi_n(a, y, b) = gb$. Now it is easy to check that the pullback of $g_\bullet : K(B, n) \to K(C, n)$ along $(g^*\alpha_\bullet)\varphi_\bullet$ in dimension n is $(K_\#(A, n) \times \mathrm{COSK}^{n-1}(E_\bullet) \times (K(A, n))_n$. A unique map $\psi_\bullet$ is thus determined and so it follows that $\delta_n g^* = 0$.

Finally we will prove that $\ker(\delta_n) \subsetneq \mathrm{im}(g^*)$. Assume $\alpha_\bullet : E_\bullet \to K(C, n)$ is given and

$\delta_n[E_\bullet]=0$. Then there is a torsor map $\varphi_\bullet: E'_\bullet \to \delta_n E_\bullet$ in $\mathrm{TORS}^{n+1}(X;\, A)$ where $E'_\bullet$ is quasi-split. Consider diagram (33) in $\mathrm{Hypgpd}_n(\mathscr{C})$:



(33)

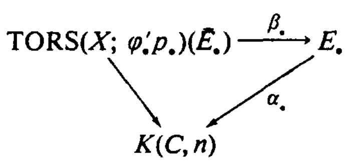
In this diagram, $\varphi'_\bullet$ is determined by $\varphi_\bullet$ on the $(n-1)$-truncation, $\psi_\bullet$ is the restriction of $\varphi_\bullet$ to the fibres $G_\bullet(E'_\bullet)$ of $E'_\bullet$ and $G_\bullet(\delta_n E_\bullet)$ of $\delta_n E_\bullet$ and $p_\bullet: G_\bullet(E'_\bullet) \to \mathrm{COSK}^{n-1}(E'_\bullet)$ is the canonical projection. In dimension n

$$p_n = \mathrm{proj.}: \Delta^\bullet(n)(E'_\bullet) \times A \longrightarrow \Delta^\bullet(n)(E'_\bullet), \quad s_n = (1,0).$$

Thus $p_\bullet$ is split by $s_\bullet$. Let $G'_\bullet = \mathrm{COSK}^{n-1}(E'_\bullet)$ for brevity. Regard $G'_\bullet$ as a torsor under itself using $1_\bullet: G'_\bullet \to G'_\bullet$ and consider the torsor

$$\bar{E}_\bullet = \mathrm{TORS}(X;\, s_\bullet)(G'_\bullet) \in \mathrm{TORS}(X;\, G_\bullet(E'_\bullet)).$$

By extending the structural hypergroupoid along $\varphi'_\bullet p_\bullet$ we have



where $\beta_\bullet$ is *both* a hypergroupoid action *and* an n-torsor map. Thus

$$[\mathrm{TORS}(X;\, \varphi'_\bullet p_\bullet)(\bar{E}_\bullet)] = [E_\bullet]$$

in $\mathrm{TORS}^n[X;\, C]$.

Similarly, by extending the structural hypergroupoid along $\alpha'_\bullet \psi_\bullet$ we have

$$E''_\bullet = \mathrm{TORS}(X;\, \alpha'_\bullet \psi_\bullet)(\bar{E}_\bullet) \in \mathrm{TORS}^n(X;\, B).$$

But $g_\bullet(\alpha'_\bullet \psi_\bullet) = (\alpha_\bullet \varphi'_\bullet) p_\bullet$ and so

$$g^*([E''_\bullet]) = [\mathrm{TORS}(X;\, \varphi'_\bullet p_\bullet)(\bar{E}_\bullet)] = [E_\bullet].$$

This proves $\ker(\delta_n) \subseteq \mathrm{im}(g^*)$. $\square$

Theorem 6.3.3.

$$\mathrm{TORS}^n[X;\,A] \xrightarrow{f^*} \mathrm{TORS}^n[X;\,B] \xrightarrow{g^*} \mathrm{TORS}^n[X;\,C]$$

is exact.

Proof. First the case $n=1$. To show $\mathrm{im}(f^*)\subseteq \ker(g^*)$ consider $g^*f^*=(gf)^*=0^*$. If $E'_\bullet=0^*E_\bullet$ we have (from diagram (3) in the proof of Theorem 2.4.1) the diagram

$$E_0\times A\times C \underset{D_1}{\overset{D_0}{\rightrightarrows}} E_0\times C \longrightarrow E'_0$$

where $D_0(y,a,c)=(y,c)$ and $D_1(y,a,c)=(ya,-0a+c)=(ya,c)$. The projection $\mathrm{pr}_C\colon E_0\times C\to C$ satisfies $\mathrm{pr}_C\,D_0=\mathrm{pr}_C\,D_1$ and induces the factorization $E'_\bullet\to K_\#(C,1)\to K(C,1)$. This shows $[E'_\bullet]=0$. Next, to show $\ker(g^*)\subseteq \mathrm{im}(f^*)$, let $E_\bullet\in \mathrm{TORS}^1(X;\,B)$ and assume $g^*E_\bullet=E'_\bullet$ is split. E'_0 appears in the exact sequence

$$E_0\times B\times A \rightrightarrows E_0\times B \longrightarrow E'_0$$

and one has a map $w'_\bullet\colon E'_\bullet\to K_\#(C,1)$ since $E'_\bullet$ is assumed split. The map $w_0\colon E_0\to C$ defined by $w_0y=w'_0q(y,0)$ yields a pullback of torsors under A:

$$\begin{array}{ccc}
E_0^*\times A & \longrightarrow & B\times A\\
\downarrow\downarrow & & \downarrow\downarrow\\
E_0^* & \longrightarrow & B\\
\downarrow & & \downarrow\\
E_0 & \xrightarrow[w_0]{} & C
\end{array}$$

There is a principal action of B on E_0^* defined by $(y,b)b'=(yb',b+b')$. (Note that $w_0(yb')=w_0(y)+gb'=gb+gb'$ so that $(yb',b+b')\in E_0^*$.) If we set $r\colon E_0^*\to E_0''$ to be the coequalizer of $E_0^*\times B\rightrightarrows E_0^*$ then the actions of A and B on E_0^* fit into a commutative diagram with exact rows:

$$\begin{array}{ccccc}
E_0^*\times A\times B & \rightrightarrows & E_0^*\times A & \longrightarrow & E_0''\times A\\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow\\
E_0^*\times B & \rightrightarrows & E_0^* & \xrightarrow[r]{} & E_0''\\
\downarrow & & \downarrow & & \downarrow\\
E_0\times B & \rightrightarrows & E_0 & \xrightarrow[p]{} & X
\end{array}$$

The left column is

$$(E_0^* \times A \rightrightarrows E_0^* \longrightarrow E_0) \times B$$

and the top row is

$$(E_0^* \times B \rightrightarrows E_0^* \longrightarrow E_0'') \times A.$$

It then follows that the right-most column is exact and thus is the 1-truncation of a torsor $E_\bullet''$ under A. Now apply f^* to $E_\bullet''$. There is a map of exact sequences

$$\begin{array}{ccc} E_0^* \times A & \longrightarrow & E_0'' \times A \times B \\ \downarrow\downarrow & & \downarrow\downarrow \\ E_0^* & \longrightarrow & E_0'' \times B \\ \downarrow & & \downarrow \\ E_0 & \longrightarrow & (f^*E_\bullet'')_0 \end{array}$$

determined by $E_0^* \to E_0''$, $(y,b) \mapsto (r(y,b),b)$ where the right column is from the extension of the structural groupoid construction. (To check that this works, note that the action of A on E_0^* is defined by $(y,b)a = (y, -fa+b)$). The map in the quotient is a torsor map $E_\bullet \to f^*E_\bullet''$ and this completes the proof that $\ker(g^*) \subseteq \mathrm{im}(f^*)$.

Now consider the case $n>1$. Suppose $g^*[E_\bullet]=0$. By Corollary 5.7.7 and Lemma 6.1.1 we may choose a representative of $[E_\bullet]$, $E_\bullet$ itself say, such that $g^*E_\bullet = E_\bullet'$ is quasi-split. If we regard these n-torsors as 1-torsors in $\mathrm{Hypgpd}_{n-1}(\mathscr{C})$ then the $n=1$ case applies and it follows that $E_\bullet = f^*E_\bullet''$ for $E_\bullet''$ obtained as in that case. This proves $\ker(g^*) \subseteq \mathrm{im}(f^*)$ and completes the proof of the theorem. □

7. Connections with classical theories

7.1. Yoneda's theory of Ext

Let $\mathscr{C}$ be an abelian category. A cohomology class in $\mathrm{Ext}^n(X,A)$ is represented by an n-fold extension of X by A,

$$0 \longrightarrow A \longrightarrow N_{n-1} \longrightarrow \cdots \longrightarrow N_0 \longrightarrow X \longrightarrow 0.$$

A map of such extensions is a commutative ladder whose 'rungs' point in the same direction and which has identity maps at X and A. This yields a category which we will denote *n-fold*(X,A) whose connected components are the elements of $\mathrm{Ext}^n(X,A)$. See [16, Chapter VII].

The well-known Dold–Kan equivalence [5, 12] between $\mathrm{Simpl}(\mathscr{C})$ and (positive) chain complexes of $\mathscr{C}$ restricts to an equivalence between $\mathrm{TORS}^n(X; A)$ and *n-fold*(X, A). Here are the details.

Let $E_\bullet$ be augmented over X and, for $1 \le i \le j$, set

$$N_j^i(E_\bullet) = E_j \cap \ker(d_0) \cap \ker(d_1) \cap \cdots \cap \ker(d_{i-1}).$$

Abbreviate $N_j^i(E_\bullet)$ by N_j^i and $N_j^j(E_\bullet)$ by $N_j(E_\bullet)$ or N_j. There is a functor $N: \mathrm{Simpl}(\mathscr{C}) \to \mathrm{Simpl}(\mathscr{C})$ defined by $N(E_\bullet)_m = N_{m+1}^1$ whose face maps $d_i': N(E_\bullet)_{m+1} \to N(E_\bullet)_m$ are the restrictions of $d_{i+1}: E_{m+2} \to E_{m+1}$ (and degeneracies $s_j' = s_{j+1}$ similarly). N^k denotes the k-th iterate of N. Diagram (34) summarizes the relationship between $E_\bullet, N(E_\bullet), N^2(E_\bullet)$, etc. The face maps of $N^k(E_\bullet)$ are shown with the subscripts of the maps of which they are the restriction.

$$\begin{array}{ccccccccccccc}
X & \longleftarrow & E_0 & \overset{d_1}{\underset{d_0}{\leftleftarrows}} & E_1 & \overset{d_2}{\underset{d_0}{\leftleftarrows}} & E_2 & \overset{d_3}{\underset{d_0}{\leftleftarrows}} & E_3 & \overset{d_4}{\underset{d_0}{\leftleftarrows}} & E_4 & & E_\bullet \\
\uparrow 1 & & \uparrow 1 & & \cup & & \cup & & \cup & & \cup & & \\
X & \longleftarrow & N_0 & \underset{d_1}{\longleftarrow} & N_1 & \overset{d_2}{\underset{d_1}{\leftleftarrows}} & N_2^1 & \overset{d_3}{\underset{d_1}{\leftleftarrows}} & N_3^1 & \overset{d_4}{\underset{d_1}{\leftleftarrows}} & N_4^1 & & N(E_\bullet) \\
\uparrow 1 & & \uparrow 1 & & \uparrow 1 & & \cup & & \cup & & \cup & & \\
X & \longleftarrow & N_0 & \underset{d_1}{\longleftarrow} & N_1 & \underset{d_2}{\longleftarrow} & N_2 & \overset{d_3}{\underset{d_2}{\leftleftarrows}} & N_3^2 & \overset{d_4}{\underset{d_2}{\leftleftarrows}} & N_4^2 & & N^2(E_\bullet)
\end{array} \qquad (34)$$

The chain complex $0 \leftarrow X \leftarrow N_0 \leftarrow N_1 \leftarrow \cdots$ is called the *Moore normal complex;* we will denote it $N^\infty(E_\bullet)$.

Observe that the short exact sequence $0 \to N_j^{i+1} \to N_j^i \xrightarrow{d_i} N_{j-1}^i \to 0$ is split by $s_i: N_{j-1}^i \to N_j^i$. Thus $N_j^i = N_j^{i+1} \oplus N_{j-1}^i$ and $E_m = N_m^1 \oplus E_{m-1}$ may be decomposed inductively into a direct sum of N_i's. Also, the face and degeneracy maps of $E_\bullet$ can be expressed in terms of the differentials in $N^\infty(E_\bullet)$. (The precise details are not required here.) This observation establishes the Dold–Kan equivalence.

Lemma 7.1.1. *$E_\bullet$ is aspherical iff $N(E_\bullet)$ is aspherical.*

Proof. For each n one has the following commutative diagram of exact sequences where $K = \Delta^\bullet(n)(E_\bullet)$, $K' = \Delta^\bullet(n-1)(N(E_\bullet))$ and where the left-hand square is a pushout:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_n^1 & \longrightarrow & E_n & \longrightarrow & E_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & E_{n-1} & \longrightarrow & 0
\end{array}$$

If $N(E_\bullet)$ is aspherical then $N_n^1 \to K'$ is epic and hence so is its pushout $E_n \to K$.

Conversely, if $E_\bullet$ is aspherical then $E_n \to K$ is epic and hence so is $N_n^1 \to K'$ by a standard abelian category diagram chase. □

Lemma 7.1.2. *Let $E_\bullet$ be augmented over X. Then $E_\bullet$ is aspherical at dimension 0 iff*

$$N_1 \xrightarrow{d_1} N_0 \xrightarrow{p} X$$

is exact (note $N_0 = E_0$).

Proof. The maps in question are

$$E_1 \xrightarrow[d]{} K \underset{p_1}{\overset{p_0}{\rightrightarrows}} E_0 \xrightarrow[p]{} X$$

where (p_0, p_1) is the kernel pair of p and d is the canonical projection defined by $dz = (d_0 z, d_1 z)$. By definition, $E_\bullet$ is aspherical at dimension 0 iff d is epic. Assume d is epic and $py = 0$. Then $(0, y) \in K$ and $(0, y) = dz$ for some $z \in E_1$. Then $z \in N_1$ and $d_1 z = y$, thus showing $\mathrm{im}(d_1) = \ker(p)$. Conversely, if $\ker(p) = \mathrm{im}(d_1)$ and $(y_0, y_1) \in K$, then $y_1 - y_0 \in \ker(p)$ so that $y_1 - y_0 = d_1 z$ for some $z \in N_1$. Then $(y_0, y_1) = d(z + s_0 y_0)$, whence d is epic. □

Corollary 7.1.3. *Let $E_\bullet$ be augmented over X. Then $E_\bullet$ is aspherical iff $N^\infty(E_\bullet)$ is exact.*

Proof. By induction: $E_\bullet$ is aspherical at dimension n iff $N(E_\bullet)$ is aspherical at dimension $n-1$ iff $N^\infty(N(E_\bullet)) = N^\infty(E_\bullet)$ is exact at $N_{n-1}(N^1(E_\bullet)) = N_n(E_\bullet)$. □

Lemma 7.1.4. *If $E_\bullet = \mathrm{COSK}^m(E_\bullet)$ then $N_{m+2}(E_\bullet) = 0$.*

Proof. An element of N_{m+2} is a matrix in $\Delta^\bullet(m+2)(E_\bullet)$ whose first $m+2$ rows consist of zeros. But then the bottom row must also consist of zeros. □

Theorem 7.1.5. *$E_\bullet \in \mathrm{TORS}^n(X; A)$ iff $N^\infty(E_\bullet) \in n\text{-}fold(X, A)$.*

Proof. If $E_\bullet$ is an n-torsor then

$$(f, d): 0 \to A \to E_{n-1} \to \Delta^\bullet(m-1)(E_\bullet) \to 0$$

is exact. Thus $y = fa$ iff $d_i y = 0$, $0 \le i \le n-1$. It follows that $0 \to A \to N_{n-1} \to N_{n-2}$ is exact. Thus, using that $E_\bullet$ is aspherical, $N^\infty(E_\bullet)$ is the exact sequence

$$0 \to A \to N_{n-1} \to \cdots \to N_0 \to X \to 0$$

in $n\text{-}fold(X, A)$.

Conversely, let the n-fold extension $0 \to A \to N_{n-1} \to \cdots \to N_0 \to X \to 0$ be given, and let $E_\bullet$ augmented over X be the corresponding simplicial object determined by the

Dold–Kan equivalence. $E_\bullet$ is aspherical by Corollary 7.1.3. The sequence

$$0 \to A \to E_{n-1} \to \Delta^\bullet(n-1)(E_\bullet) \to 0$$

is exact where $(A \to E_{n-1}) = (A \to N_{n-1} \to E_{n-1})$. Similarly,

$$0 \to 0 \to E_m \to \Delta^\bullet(m)(E_\bullet) \to 0$$

is exact by the same argument in dimensions $m \geq n$, thus showing $E_\bullet = \mathrm{COSK}^{n-1}(E_\bullet)$. We must find an appropriate $(n-1)$-dimensional hypergroupoid structure on E_{n-1} in order to show that $E_\bullet$ is a torsor. First note that $A = N_n(E_\bullet)$ is a direct summand of E_n. Let $\alpha: E_n \to A$ be the projection. Since $A = \{ y \in E_n \mid d_j y = 0, 0 \leq j \leq n-1\}$, an element of A may be represented by $(0, \ldots, 0, a, 0, \ldots, 0)$ where all but one component is 0. Observe that if $y \in E_n$ and $\alpha y = a$, then y decomposes as

$$(y_0, \ldots, y_i, \ldots, y_n) = (y_0, \ldots, y_i + (-1)^{n-i}a, \ldots, y_n) - (0, \ldots, (-1)^{n-i}a, \ldots, 0).$$

A hypergroupoid structure on E_{n-1} may then be obtained as follows: given $(y_0, \ldots, -, \ldots, y_n) \in \Lambda^i(n)(E_\bullet)$, choose as $y_i \in E_{n-1}$ to fill the open component. Define $\Lambda^i(n)(E_\bullet) \to E_{n-1}$ by

$$(y_0, \ldots, -, \ldots, y_n) \mapsto y_i + (-1)^{n-1}\alpha(y_0, \ldots, y_i, \ldots, y_n).$$

It is easy to check that this definition is independent of the choice of y_i by using the decomposition of y as above together with the fact that any two choices of 'y_i' have the same faces and thus differ by an element of A. These maps determine a hypergroupoid structure, Fib, on E_{n-1} and one has a monic $K(A, n-1) \to \mathrm{Fib}$ defined in dimension $n-1$ by $A \to E_{n-1}$ and in dimension n by

$$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{n-1}, -) \in \Lambda^n(n)(E_\bullet).$$

The short exact sequence of $(n-1)$-dimensional hypergroupoids

$$0 \to K(A, n-1) \to \mathrm{Fib} \to \mathrm{COSK}^{n-2}(E_\bullet) \to 0$$

establishes that $E_\bullet$ is an n-torsor under A whose fiber is Fib. □

It is clear that torsor maps correspond to maps of n-fold extensions so that $\mathrm{TORS}^n[X; A] = \mathrm{Ext}^n(X, A)$. The correspondence between the group structures reduces to a verification in dimension 1 which is routine and will be omitted.

7.2. Comonad cohomology

Let $\mathscr{C}$ be monadic over $\mathscr{S}$ (sets) and denote the associated adjoint pair $F, U: \mathscr{C} \to \mathscr{S}$. The functor $G = FU: \mathscr{C} \to \mathscr{C}$ together with natural transformations obtained from the unit and co-unit of the adjunction determine an augmented simplicial object

$$G^\bullet X = (X \leftarrow GX \leftleftarrows G^2X \cdots)$$

called the *standard resolution* of $X_\bullet$. Given an abelian group object A of $\mathscr{C}$, there is a cochain complex defined by $C^n(X,A)=\mathscr{C}(G^{n+1}X,A)$ and $\partial_n:C^n\to C^{n+1}$ the alternating sum of the maps induced by the face operators $G^{n+2}X\to G^{n+1}X$. The homology groups of this complex, denoted $H^*_G(X;A)$, are the *comonad cohomology groups of X with co-efficients in A relative to the comonad G.*

Duskin showed how to represent an element of $H^n_G(X;A)$ as a '$K(A,n)$-torsor' and did so in the more general case where $\mathscr{S}$ can be replaced by any finitely complete category [7]. The concept of a '$K(A,n)$-torsor' is the immediate predecessor of the concept of torsor defined in this paper. We will show that the two are almost the same (actually coinciding in many examples) and thus relate the groups $\mathrm{TORS}^n[X;A]$ to those classical cohomology theories which were earlier shown to coincide with comonad cohomology groups. See [3].

The functor $U:\mathscr{C}\to\mathscr{S}$ creates limits and coequalizers of U-contractible pairs. (See [13, Chapter VI] for details). It follows easily that $\mathscr{C}$ is exact and that $E_\bullet\in\mathrm{TORS}^n(X;A)$ in $\mathscr{C}$ iff $UE_\bullet\in\mathrm{TORS}^n(UX;UA)$ in $\mathscr{S}$. Since every 1-torsor in $\mathscr{S}$ is split, $UE_{n-1}=U\Delta^\bullet(n-1)(E_\bullet)\times UA$ where UA acts on UE_{n-1} by translation on the right-hand factor. Note also that $UE_\bullet$ is split as a simplicial set (and $E_\bullet$ is then said to be *U-split*) because it is aspherical (Lemma 1.8.2). The definition in [7, p. 66] is, in effect, that $E_\bullet\to K(A,n)$ is '$K(A,n)$-torsor rel U' if $E_\bullet$ is U-split, $E_\bullet=\mathrm{COSK}^{n-1}(E_\bullet)$ and $UE_\bullet\to K(UA,n)$ is an n-dimensional hypergroupoid action. A map of $K(A,n)$-torsors is one which is equivariant and which preserves the U-splittings. We will denote the resulting category $\mathrm{TORS}^n_U(X;A)$. These observations show:

Proposition 7.2.1. $\mathrm{TORS}^n(X;A)$ *is a subcategory of* $\mathrm{TORS}^n_U(X;A)$. □

The standard resolution $G^\bullet X$ is U-split and has the following universal property: if $E_\bullet$ is any U-split simplicial object augmented over X, then there is a uniquely determined simplicial map $G^\bullet X\to E_\bullet$ which preserves the U-splitting. Hence, given any $E_\bullet\in\mathrm{TORS}^n_U(X;A)$, the $(n-2)$-truncation of the unique $G^\bullet X\to E_\bullet$ determines by Proposition 3.8.1, a $K(A,n)$-torsor map $E'_\bullet\to E_\bullet$ where $\mathrm{TR}^{n-2}(E'_\bullet)=\mathrm{TR}^{n-2}(G^\bullet X)$. $E'_\bullet$ is called the *standard torsor associated with* $E_\bullet$. Duskin established his interpretation bijections [7, Chapter 8] correlating $K(A,n)$-torsors with n-cocycles by use of the standard torsor. Further, given an abelian group object homomorphism $f:A\to B$, the functor $\mathrm{TORS}^n_U[X;f]$ induced by extension-of-the-structural-group corresponds to the functor $H^n_G(X;f)$ induced by composing f with n-cocycles.

Proposition 7.2.2. *The induced map* $\mathrm{TORS}^n[X;A]\to\mathrm{TORS}^n_U[X;A]$ *is a monomorphism.* □

Remark. A $K(A,n)$-torsor map is, in particular, a torsor map. That is why no collapsing occurs in $\mathrm{TORS}^n[X;A]\to\mathrm{TORS}^n_U[X,A]$.

The distinction between $K(A,n)$-torsors and n-torsors is that the former need not be aspherical. Nevertheless, examples where $\mathrm{TORS}^n[-;-]=\mathrm{TORS}^n_U[-;-]$ abound.

Lemma 7.2.3. *If the simplicial set $E_\bullet$ is split and is a Kan complex, then it is aspherical.*

Proof. Given $(x_0, \dots, x_n) \in \Delta^\bullet(n)(E_\bullet)$, then $(s_n x_0, \dots, s_n x_n, -) \in \Lambda^{n+1}(n+1)(E_\bullet)$ and there exists $z \in E_{n+2}$ such that $d_i z = s_n x_i$, $0 \le i \le n$, because $E_\bullet$ is a Kan complex. Then $x_0, \dots, x_n$ comprise the faces of $d_{n+2} z$. □

Every simplicial group is a Kan complex [15]. Hence U-split simplicial groups are aspherical.

Corollary 7.2.4. $\mathrm{TORS}^n = \mathrm{TORS}^n_U$ *if $\mathscr{C}$ is a category monadic over $\mathscr{S}$ whose objects have an underlying group structure.* □

In order for coincidence to occur, it is not necessary that every simplicial object in $\mathscr{C}$ be a Kan complex. For example, $K(A,n)$-torsors in the category of G-sets, $\mathscr{S}^G$ (G a group) need not be Kan complexes. However, the standard resolution, $X \times \mathrm{DEC}(U(G), 1)$, is a Kan complex. Thus every $K(A,n)$-torsor $E_\bullet$ is mapped into by a torsor (the associated standard torsor) and it follows easily that $\mathrm{TORS}^n[X; A] \to \mathrm{TORS}^n_U[X; A]$ is an isomorphism.

For another example, Duskin pointed out in [6] that if the objects of $\mathscr{C}$ admit a 'Mal'cev operation' (a ternary operation W satisying $W(x,x,y) = W(y,x,x) = y$) then the conclusion of Corollary 7.2.4 still holds. The reason is that the standard resolution of a Mal'cev algebra is aspherical (see [17, Proposition 612]) so that standard $K(A,n)$-torsors are torsors in the sense of this paper. Any group has a Mal'cev operation defined by $W(x,y,z) = xy^{-1}z$.

7.3. Sheaf cohomology

Let $\mathscr{E}$ be an arbitrary topos. The category $\mathrm{Ab}(\mathscr{E})$ of abelian group objects of $\mathscr{E}$ is an abelian category. If $\mathrm{Ab}(\mathscr{E})$ has enough injectives (as is the case when $\mathscr{E}$ is a Grothendieck topos), then the derived functors of $\mathscr{E}(1,-): \mathrm{Ab}(\mathscr{E}) \to \mathrm{Ab}$ are the *cohomology groups,* $H^*(\mathscr{E}, -)$, of $\mathscr{E}$. Now, regardless of whether or not $\mathrm{Ab}(\mathscr{E})$ has enough injectives, one may consider the cohomology groups $\mathrm{TORS}^*_{\mathscr{E}}[1; -]$. We will show that if $\mathscr{E}$ is a Grothendieck topos then $\mathrm{TORS}^*_{\mathscr{E}} = H^*$. The proof consists of showing that $\mathrm{TORS}^n_{\mathscr{E}}$ vanishes on injectives [9,4]. A Grothendieck topos $\mathscr{E}$ has a 'free abelian group object' functor $F: \mathscr{E} \to \mathrm{Ab}(\mathscr{E})$ left adjoint to the forgetful functor $U: \mathrm{Ab}(\mathscr{E}) \to \mathscr{E}$. We will use that $H^n(\mathscr{E}, -) = \mathrm{Ext}^n_{\mathrm{Ab}(\mathscr{E})}(Z, -)$ where $Z = F(1)$. See [11, Chapter 8] for details.

Proposition 7.3.1. $\mathrm{Ext}^1_{\mathrm{Ab}(\mathscr{E})}(Z, A) = \mathrm{TORS}^1_{\mathscr{E}}[1; A]$.

Proof (following Johnstone [11]). An element $(f,g): 0 \to A \to E \to Z \to 0$ of Ext^1 yields a 1-torsor $E \times A \rightrightarrows E \to Z$ in $\mathrm{Ab}(\mathscr{E})$ where the action of A on E is defined by $ya = -fa + y$. If this torsor is pulled back along the unit of the adjunction evaluated at $1, 1 \to Z$, one obtains a 1-torsor in $\mathrm{TORS}^1_{\mathscr{E}}(1; A)$.

Conversely, given $E_\bullet \in \mathrm{TORS}^1_{\mathscr{E}}(1; A)$, define $nE_\bullet = E_\bullet \otimes \cdots \otimes E_\bullet$ (n times) if $n > 0$, set $nE_\bullet = (-n)(-E_\bullet)$ if $n < 0$, and $0E_\bullet =$ the trivial torsor over 1. Let $nE_0 = (nE_\bullet)_0$. Define E to be $\coprod_{\text{all } n} nE_0$. E has an abelian group operation $E \times E \to E$ induced on summands by the maps $nE_0 \times mE_0 \to (n+m)E_0$. An epimorphism $E \to Z = \coprod_{\text{all } n} 1$ is induced by $nE_0 \to \coprod_n 1$. The kernel of this map is $0E_0 = A$ and thus the torsor $E_\bullet$ determines the short exact sequence $0 \to A \to E \to Z \to 0$ and $E_0 \to E \to Z = E_0 \to 1 \to Z$ is clearly a pullback. □

Given $X \in \mathscr{E}$, the functor $X^*: \mathscr{E} \to \mathscr{E}/X$ is defined by $X^*Y = \mathrm{pr}: Y \times X \to X$.

Lemma 7.3.2. *X^* preserves injective abelian group objects.*

A proof is given in *Théorie des Topos et Cohomologie Etale des Schémas* SGA 4, IV, Proposition 11.3.1, pp. 498–499.

Theorem 7.3.3. $\mathrm{TORS}^1_{\mathscr{E}}[X; -]$ *vanishes on injectives.*

Proof. The conclusion follows from Proposition 7.3.1 if $X = 1$. Otherwise,

$$\mathrm{TORS}^1_{\mathscr{E}}(X; I) = \mathrm{TORS}^1_{\mathscr{E}/X}(1; X^*I) = \mathrm{Ext}^1_{\mathrm{Ab}(\mathscr{E}/X)}(Z; X^*I),$$

again by Proposition 7.3.1. Since X^*I is injective if I is, the theorem follows. □

Corollary 7.3.4. $\mathrm{TORS}^n_{\mathscr{E}}[1; -] = H^n(\mathscr{E}, -)$ *if $\mathscr{E}$ is a Grothendieck topos.*

Proof. If I is injective then the attached 1-torsor of any $E_\bullet \in \mathrm{TORS}^n_{\mathscr{E}}(1; I)$ is split, by Theorem 7.3.3. Thus $\mathrm{TORS}^n_{\mathscr{E}}[1; -]$ vanishes on injectives and the isomorphism follows. □

References

[1] M. Barr, Embedding of categories, Proc. Amer. Math. Soc. 37 (1978) 42–46.

[2] M. Barr, Exact categories, in: M. Barr, P.A. Grillet, D.H. Van Osdol, Exact Categories and Categories of Sheaves, Lecture Notes in Mathematics, Vol. 236 (Springer, Berlin–New York, 1971).

[3] M. Barr and J. Beck, Homology and standard constructions, in: B. Eckmann, Ed., Seminar on Triples and Categorical Homology Theory, Lecture Notes in Mathematics, Vol. 80 (Springer, Berlin–New York, 1969).

[4] H. Cartan and S. Eilenberg, Homological Algebra (Princeton University Press, Princeton, NJ, 1956).

[5] A. Dold, Homology of symmetric products and other functors of complexes, Ann. of Math. 68 (2) (1958) 54–80.

[6] J.W. Duskin, Higher dimensional torsors and the cohomology of topoi: the abelian theory, in: M.P. Fourman, C.J. Mulvey and D.S. Scott, Eds., Applications of Sheaves, Proceedings, Durham 1977, Lecture Notes in Mathematics, Vol. 753 (Springer, New York, 1979).

[7] J.W. Duskin, Simplicial methods and the interpretation of 'triple' cohomology, Mem. Amer. Math. Soc. Vol. 3, Issue 2, No. 163 (American Mathematical Society, Providence, RI, 1975).

[8] P. Glenn, Realization of cohomology classes by torsors under hypergroupoids, Thesis, State University of New York at Buffalo, Buffalo, New York (1977).

[9] A. Grothendieck, Sur quelque points d'algèbre homologique, Tohoku Math. J. 9 (1957) 119–221.

[10] A. Grothendieck, Techniques de construction en géométrie analytique, IV. Formalisme general des foncteurs representables, in: Seminaire Henri Cartan, 1960/61, No. 11 (Ecole Normale Superieure, Paris, 1961).

[11] P.T. Johnstone, Topos Theory (Academic Press, New York, 1977).

[12] D.M. Kan, Functors involving c.s.s. complexes, Trans. Amer. Math. Soc. 87 (1958) 330–346.

[13] S. MacLane, Categories for the Working Mathematician (Springer, New York, 1971).

[14] S. MacLane, Homology (Springer, New York, 1963).

[15] J.P. May, Simplicial Objects in Algebraic Topology (Van Nostrand, Princeton, New Jersey, 1967).

[16] B. Mitchell, Theory of Categories (Academic Press, New York, 1965).

[17] J.D. Smith, Mal'cev Varieties, Lecture Notes in Mathematics, Vol. 554 (Springer, Berlin–New York, 1976).

[18] N. Steenrod, The Topology of Fibre Bundles (Princeton University Press, Princeton, New Jersey, 1951).